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# THE GENERALIZED CONTINUUM HYPOTHESIS REVISITED

BY

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ABSTRACT

We can reformulate the generalized continuum problem as: for regular $\kappa < \lambda$ we have $\lambda$ to the power $\kappa$ is $\lambda$, We argue that the reasonable reformulation of the generalized continuum hypothesis, considering the known independence results, is "for most pairs $\kappa < \lambda$ of regular cardinals, $\lambda$ to the revised power of $\kappa$ is equal to $\lambda$". What is the revised power? $\lambda$ to the revised power of $\kappa$ is the minimal cardinality of a family of subsets of $\lambda$ each of cardinality $\kappa$ such that any other subset of $\lambda$ of cardinality $\kappa$ is included in the union of strictly less than $\kappa$ members of the family. We still have to say what "for most" means. The interpretation we choose is: for every $\lambda$, for every large enough $\kappa < \beth_\omega$. Under this reinterpretation, we prove the Generalized Continuum Hypothesis.

## ANNOTATED CONTENTS



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## 0. Introduction

I had a dream, quite a natural one for a mathematician in the twentieth century: to solve a Hilbert problem, preferably positively. This is quite hard for (at least) three reasons:

(a) those problems are almost always hard,

(b) almost all have been solved,

(c) my (lack of) knowledge excludes almost all.

Now (c) points out the first Hilbert problem as it is in set theory; also, being the first, it occupies a place of honor.

The problem asks "is the continuum hypothesis true?", i.e.,

(1) is $2^{\aleph_0} = \aleph_1$?

More generally, is the generalized continuum hypothesis true? Which means:

(2) is $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all ordinals $\alpha$?

I think the meaning of the question is: what are the laws of cardinal arithmetic? It was known that addition and multiplication of infinite cardinals is "trivial", i.e., previous generations have not left us anything to solve:

$$\lambda + \mu = \lambda \times \mu = \max\{\lambda, \mu\}.$$

This would have certainly made elementary school pupils happier than the usual laws, but we have been left with exponentiation only. As there were two operations on infinite cardinals increasing them — $2^\lambda$ and $\lambda^+$ — it was most natural to assume that those two operations are the same; in fact, in this case also exponentiation becomes very simple; usually $\lambda^\mu = \max\{\lambda, \mu^+\}$, the exception being that when $\mathrm{cf}(\lambda) \leq \mu < \lambda$ we have $\lambda^\mu = \lambda^+$, where

$$\mathrm{cf}(\lambda) =: \min\{\kappa\colon \text{ there are } \lambda_i < \lambda \text{ for } i < \kappa \text{ such that } \lambda = \sum_{i<\kappa} \lambda_i\}.$$

Non-set theorists may be reminded that $\lambda = \mu^+$ if $\mu = \aleph_\alpha$ and $\lambda = \aleph_{\alpha+1}$, and then $\lambda$ is called the successor of $\mu$ and we know $\mathrm{cf}(\aleph_{\alpha+1}) = \aleph_{\alpha+1}$; we call a cardinal $\lambda$ regular if $\mathrm{cf}(\lambda) = \lambda$ and singular otherwise. So successor cardinals, are regular and also $\aleph_0$, but it is "hard to come by" other regular cardinals, so we may ignore them. Note $\aleph_\omega = \sum_{n<\omega} \aleph_n$ is the first singular cardinal, and for $\delta$ a limit ordinal $> |\delta|$ we have $\aleph_\delta$ singular, but there are limit $\delta = \aleph_\delta$ for which $\aleph_\delta$ is singular.

Probably the interpretation of Hilbert's first problem as "find all laws of cardinal arithmetic" is too broad[1], still "is cardinal arithmetic simple" is a reasonable interpretation.

Unfortunately, there are some "difficulties". On the one hand, Gödel had proved that GCH may be true (specifically it holds in the universe of constructible sets, called $L$). On the other hand, Cohen had proved that CH may be false (by increasing the universe of sets by forcing); in fact, $2^{\aleph_0}$ can be anything reasonable, i.e., $\mathrm{cf}(2^{\aleph_0}) > \aleph_0$.

Continuing Cohen, Solovay proved that $2^{\aleph_n}$ for $n < \omega$ can be anything reasonable: it should be non-decreasing and $\mathrm{cf}(2^\lambda) > \lambda$. Continuing this, Easton proved that the function $\lambda \mapsto 2^\lambda$ for regular cardinals is arbitrary (except for the laws above). Well, we can still hope to salvage something by proving that (2) holds for "most" cardinals; unfortunately, Magidor had proved the consistency of $2^\lambda > \lambda^+$ for all $\lambda$ in any pregiven initial segment of the cardinals and then Foreman and Woodin [FW] for all $\lambda$.

Such difficulties should not deter the truly dedicated ones; first note that we should not identify exponentiation with the specific case of exponentiation $2^\lambda$, in fact Easton's results indicate that on this (for $\lambda$ regular) we cannot say anything more, but they do not rule out saying something on $\lambda^\mu$ when $\mu < \lambda$, and we can rephrase the GCH as

(3) for every regular $\kappa < \lambda$ we have $\lambda^\kappa = \lambda$.

Ahah, now that we have two parameters we can look again at "for most pairs

1 On this see [Sh:g] or [Sh:400a]; note that under this interpretation of the problem there is much to say.

of cardinals (3) holds". However, this is a bad division, because, say, a success for $\kappa = \aleph_1$ implies a success for $\kappa = \aleph_0$.

To rectify this we suggest another division; we define "$\lambda$ to the revised power of $\kappa$", for $\kappa$ regular $< \lambda$, as

$$\lambda^{[\kappa]} = \operatorname{Min}\Big\{|\mathcal{P}|: \mathcal{P} \text{ a family of subsets of } \lambda \text{ each of cardinality } \kappa \\ \text{such that any subset of } \lambda \text{ of cardinality } \kappa \\ \text{is contained in the union of } < \kappa \text{ members of } \mathcal{P}\Big\}.$$

This answers the criticism above and is a better slicing because:

(A) for every $\lambda > \kappa$ we have: $\lambda^\kappa = \lambda$ iff $2^\kappa \leq \lambda$ and for every regular $\theta \leq \kappa$, $\lambda^{[\theta]} = \lambda$.

(B) By Gitik and Shelah [GiSh 344], the values of, e.g., $\lambda^{[\aleph_0]}, \dots, \lambda^{[\aleph_n]}$ are essentially independent.

Now we rephrase the generalized continuum hypothesis as:

(4) for most pairs $(\lambda, \kappa)$, $\lambda^{[\kappa]} = \lambda$.

Is such a reformulation legitimate? As an argument, I can cite, from the book [Br] on Hilbert's problems, Lorentz's article on the thirteenth problem. The problem was

($*$) Prove that the equation of the seventh degree $x^7 + ax^3 + bx^2 + cx + 1 = 0$ is not solvable with the help of any continuous functions of only two variables.

Lorentz does not even discuss the change from 7 to $n$ and he shortly changes it to (see [Br, Ch. II, p. 419])

($*$)$'$ Prove that there are continuous functions of three variables not represented by continuous functions of two variables.

Then, he discusses Kolmogorov's solution and improvements. He opens the second section with ([Br, p. 421, 16–22]): "that having disproved the conjecture is not solving it, we should reformulate the problem in the light of the counterexamples and prove it, which in his case: (due to Vituvskin) the fundamental theorem of the Differential Calculus: there are $r$-times continuously differential functions of $n$ variables not represented by superpositions of $r$ times continuously times differential functions of less than $n$ variables".

Concerning the fifth problem, Gleason (who makes a major contribution to its solution) says (in [AAC90]): "Of course, many mathematicians are not aware that the problem as stated by Hilbert is not the problem that has been ultimately called the Fifth Problem. It was shown very, very early that what he was asking people to consider was actually false. He asked to show that the action of a locally-euclidean group on a manifold was always analytic, and that's false. It's only the group itself that's analytic, the action on a manifold need not be. So you had to change things considerably before you could make the statement he was concerned with true. That's sort of interesting, I think. It's also part of the way a mathematical theory develops. People have ideas about what ought to be so and they propose this as a good question to work on, and then it turns out that part of it isn't so".

In our case, I feel that while the discovery of $L$ (the constructible universe) by Gödel and the discovery of forcing by Cohen are fundamental discoveries in set theory, things which are and will continue to be in its center, forming a basis for flourishing research, and they provide for the first Hilbert problem a negative solution which justifies our reinterpretation of it. Of course, it is very reasonable to include independence results in a reinterpretation.

Back on firmer ground, how will we interpret "for most"? The simplest ways are to say "for each $\lambda$ for most $\kappa$" or "for each $\kappa$ for most $\lambda$". The second interpretation holds in a non-interesting way: for each $\kappa$ for many $\lambda$'s, $\lambda^\kappa = \lambda$ hence $\lambda^{[\kappa]} = \lambda$ (e.g. $\mu^\kappa$ when $\mu \geq 2$). So the best we can hope for is: for every $\lambda$ for most small $\kappa$'s (remember we have restricted ourselves to regular $\kappa$ quite smaller than $\lambda$). To fix the difference we restrict ourselves to $\lambda > \beth_\omega > \kappa$. Now what is a reasonable interpretation of "for most $\kappa < \beth_\omega$"? The reader may well stop and reflect. As "all" is forbidden (by [GiSh:344] even finitely many exceptions are possible), the simplest offer I think is "for all but boundedly many".

So the best we can hope for is ($\beth_\omega$ is for definiteness):

(5) if $\lambda > \beth_\omega$, for every large enough regular $\kappa < \beth_\omega, \lambda^{[\kappa]} = \lambda$ (and similarly replacing $\beth_\omega$ by any strong limit cardinal).

If the reader has agreed so far, he is trapped into admitting that here we solved Hilbert's first problem positively (see 0.1 below). Now we turn from fun to business. A consequence is

$(*)_6$ for every $\lambda \geq \beth_\omega$ for some $n$ and[2] $\mathcal{P} \subseteq [\lambda]^{<\beth_\omega}$ of cardinality $\lambda$, every $a \in [\lambda]^{<\beth_\omega}$ is the union of $< \beth_n$ members of $\mathcal{P}$.

The history above was written just to lead to (5); for a fuller history see [Sh:g].

More fully, our main result is

0.1 THE REVISED GCH THEOREM: *Assume we fix an uncountable strong limit cardinal $\mu$ (i.e., $\mu > \aleph_0, (\forall\theta < \mu)(2^\theta < \mu)$, e.g., $\mu = \beth_\omega = \sum \beth_n$ where $\beth_0 =$*

2 where $[\lambda]^{<\kappa} = \{a \subseteq \lambda : |a| < \kappa\}$.

$\aleph_0, \beth_{n+1} = 2^{\beth_n}$).

*Then for every $\lambda \geq \mu$ for some $\kappa < \mu$ we have:*

(a) $\kappa \leq \theta < \mu$ & $\theta$ *regular* $\Rightarrow \lambda^{[\theta]} = \lambda$,

(b) *there is a family $\mathcal{P}$ of $\lambda$ subsets of $\lambda$ each of cardinality $< \mu$ such that every subset of $\lambda$ of cardinality $\mu$ is equal to the union of $< \kappa$ members of $\mathcal{P}$.*

*Proof:* It is enough to prove it for singular $\mu$.

Clause (a) follows by clause (b) (just use $\mathcal{P}_\theta = \{a \in \mathcal{P} : |a| \leq \theta\}$) and clause (b) holds by 1.2(4)+1.3. ∎

In §1 we prove the theorem using a generic embedding based on [Sh:g, Ch. VI, §1] (hence using simple forcing) and give some applications; mainly, they are reformulations. For example, for $\lambda \geq \beth_\omega$ for every regular $\theta < \beth_\omega$ large enough, there is no tree with $\lambda$ nodes and $> \lambda$ $\theta$-branches. Also we explain that this is sufficient for proving that, e.g., a topology (not necessarily even $T_0$!) with a base of cardinality $\mu \geq \beth_\omega$ and $> \mu$ open sets has at least $\beth_{\omega+1}$ open sets (relying on [Sh 454a]).

In 2.1 we give another proof (so not relying on §1), more inside pcf theory and saying somewhat more. In 2.6 we show that a property of $\mu = \beth_\omega$ which suffices is: $\mu$ is a limit cardinal such that $|\mathfrak{a}| < \mu \Rightarrow |\mathrm{pcf}(\mathfrak{a})| < \mu$ giving a third proof. This is almost a converse to 2.5. Now §3 deals with applications: we show that for $\lambda \geq \beth_\omega$, $2^\lambda = \lambda^+$ is equivalent to $\diamondsuit_{\lambda^+}$ (moreover $\lambda = \lambda^{<\lambda}$ is equivalent to $(D\ell)_\lambda$, a weak version of diamond). We also deal with a general topology problem: can every space be divided into two pieces, no one containing a compactum (say a topological copy of ${}^\omega 2$), showing its connection to pcf theory, and proving a generalization when the cardinal parameter is $> \beth_\omega$. Lastly, in an appendix, we prove there are no tiny models for theories with a non-trivial type (see [LaPiRo]) of cardinality $\geq \beth_\omega$, partially solving a problem from Laskowski, Pillay and Rothmaler [LaPiRo].

For other applications see [Sh 575, §8]. This work is continued in [Sh 513]; for further discussion see [Sh 666]. For more on the general topology problem see [Sh 668].

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## 1 The generic ultrapower proof

1.1 THEOREM: *Assume $\mu$ is strong limit singular and $\lambda > \mu$. Then there are only boundedly many $\kappa < \mu$ such that for some $\theta \in (\mu, \lambda)$ we have $\mathrm{pp}_{\Gamma(\mu^+,\kappa)}(\theta) \geq \lambda$ (so $\kappa \leq \mathrm{cf}(\theta) < \mu < \theta$).*

We list some conclusions, which are immediate by older works.

*1.2 Conclusion:* For every $\mu$ strong limit such that $\mathrm{cf}(\mu) = \sigma < \mu < \lambda$, for some $\kappa < \mu$ we have:

(1) for every $\mathfrak{a} \subseteq \mathrm{Reg} \cap (\mu, \lambda)$ of cardinality $\leq \mu$ we have $\sup \mathrm{pcf}_{\kappa-\mathrm{complete}}(\mathfrak{a}) \leq \lambda$,

(2) there is no family $\mathcal{P}$ of $> \lambda$ subsets of $\lambda$ such that for some regular $\theta \in (\kappa, \mu)$ we have: $A \neq B \in \mathcal{P} \Rightarrow |A \cap B| < \theta$ & $|A| \geq \theta$

(3) $\mathrm{cov}(\lambda, \mu^+, \mu^+, \kappa) \leq \lambda$ (equivalently $\mathrm{cov}(\lambda, \mu, \mu, \kappa) \leq \lambda$ as without loss of generality $\mathrm{cf}(\kappa) > \sigma$).

Hence

(4) there is $\mathcal{P} \subseteq [\lambda]^{<\mu}$ such that $|\mathcal{P}| = \lambda$ and every $A \in [\lambda]^{\leq \mu}$ is equal to the union of $< \kappa$ members of $\mathcal{P}$,

(5) there is no tree with $\lambda$ nodes and $> \lambda$ $\theta$-branches when $\theta \in (\kappa, \mu)$ is regular.

*Proof:* By [Sh:g]; in detail (we repeat rather than quote immediate proofs).
1) Let $\kappa$ be as in 1.1. Without loss of generality $\mathrm{cf}(\lambda) \notin [\kappa, \mu)$.

Note that $\sup(\mathrm{pcf}_{\kappa\text{-complete}}(\mathfrak{a})) \leq \sup\{\mathrm{pp}_{\Gamma(|\mathfrak{a}|^+, \kappa)}(\lambda'): \lambda' = \sup(\mathfrak{a} \cap \lambda')$ and $\mathrm{cf}(\lambda') \geq \kappa$ so $\mathrm{cf}(\lambda') \leq |\mathfrak{a}| < \mu\}$, and easily the latter is $\leq \lambda$ by 1.1.
2) By part (4) it is easy (let $\mathcal{P}_4 \subseteq [\lambda]^{<\mu}$ be as in part (4) and $\theta, \mathcal{P}_2$ be a counterexample to part (2), so for every $A \in \mathcal{P}_2$ we can find $\mathcal{P}'_A \subseteq \mathcal{P}_4$ such that $|\mathcal{P}'_A| < \kappa$ and $A = \bigcup\{B: B \in \mathcal{P}'_A\}$ hence there is $B_A \in \mathcal{P}'_A$ such that $|B_A| = \theta$. So $A \mapsto B_A$ is a function from $\mathcal{P}_2$ into $\mathcal{P}_4$ and $B_A \in [A]^\theta$ and $A_1 \neq A_2 \in \mathcal{P}_2 \Rightarrow |A_1 \cap A_2| < \theta$ & $\theta \leq |A_1|$ & $\theta \leq |A_2|$ so the function is one-to-one so $|\mathcal{P}_2| \leq |\mathcal{P}_4| \leq \lambda$, contradiction).
3) By [Sh:g, Ch. II, 5.4].
4) Let $\mathcal{P}_0 \subseteq [\lambda]^{<\mu}$ be such that $|\mathcal{P}_0| \leq \lambda$ and every $A \subseteq [\lambda]^{\leq \mu}$ is included in the union of $< \kappa$ members of $\mathcal{P}_0$ (exists by part (3)). Define $\mathcal{P} = \{B$: for some $A \in \mathcal{P}_0$, $B \subseteq A\}$ so $\mathcal{P} \subseteq [\lambda]^{<\mu}$ and $|\mathcal{P}_0| \leq |\mathcal{P}_0| \cdot \sup\{2^{|A|}: A \in \mathcal{P}_0\} \leq \lambda \cdot \mu = \lambda$. Now for every $A \in [\lambda]^{\leq \mu}$ we can find $\alpha < \kappa$ and $B_i \in \mathcal{P}_0$ for $i < \alpha$ such that $A \subseteq \bigcup_{i<\alpha} B_i$. Let $B'_i = A \cap B_i$ for $i < \alpha$ so $B'_i \in \mathcal{P}$ and $A = \bigcup_{i<\alpha} B'_i$ as required.
5) Follows by part (2): if the tree is $T$, without loss of generality its set of nodes is $\subseteq \lambda$ and the set of $\theta$-branches cannot serve as a counterexample. $\blacksquare_{1.2}$

*1.3 Remark:* We can let $\mu$ be regular (strong limit $> \aleph_0$) if we restrict ourselves in 1.2(1) to $|\mathfrak{a}| < \mu$, and in 1.2(3),(4) to $A \in [\lambda]^{<\mu}$ as if for $\mu' \in \{\mu' < \mu: \mu'$ strong limit singular$\}, \kappa(\mu', \lambda)$ is as in 1.2, then by Fodor's lemma for some $\kappa = \kappa(\lambda)$ the set $S'_\kappa = \{\mu' < \mu: \kappa(\mu', \lambda) = \kappa\}$ is stationary: this $\kappa$ can serve.

The stimulation for proving this was in [Sh 454a] where we actually use:

*1.4 Conclusion:* Assume $\mu$ is strong limit, $\lambda \geq \mu$. Then for some $\kappa < \mu$ and family $\mathcal{P}$, $|\mathcal{P}| \leq \lambda$ we have: for every $n < \omega$ and $\sigma \in (\kappa, \mu)$ and $f$: $[\beth_n(\sigma)^+]^{n+1} \to \lambda$, for some $A \subseteq \beth_n(\sigma)^+$ of cardinality $\sigma^+$ we have $f \restriction A \in \mathcal{P}$.

*Proof:* Let $\kappa$ be as in 1.2 (or 1.2A), and $\mathcal{P}$ as in 1.2(4), and let

$$\mathcal{P}_1 = \{f\colon f \text{ a function from some bounded subset } A \text{ of } \mu \text{ into some } B \in \mathcal{P} \text{ (hence } |B| < \mu)\}.$$

As $\mu$ is strong limit and $|\mathcal{P}| \leq \lambda$, $\mu \leq \lambda$ clearly $|\mathcal{P}_1| \leq \lambda$. Now for any given $f$: $[\beth_n(\sigma)^+]^{n+1} \to \lambda$, we can find $\alpha < \kappa$ and $B_i \in \mathcal{P}$ for $i < \alpha$ such that $\mathrm{Rang}(f) \subseteq \bigcup_{i<\alpha} B_i$. Define $g$: $[\beth_n(\sigma)^+]^{n+1} \to \alpha$ by: $g(w) = \mathrm{Min}\{i < \alpha\colon f(w) \in B_i\}$, so by the Erdös–Rado theorem for some $A \subseteq \beth_n(\sigma)^+$, we have: $|A| = \sigma^+$ and $g \restriction A$ is constantly $i(*)$. Now $f \restriction A \in \mathcal{P}_1$ so we have finished. $\blacksquare_{1.4}$

*1.5 Conclusion:* If $\lambda = \aleph_0$ or $\lambda$ strong limit of cofinality $\aleph_0$, $(\Omega, \mathcal{T})$ is a topology (i.e. $\Omega$ the set of points, $\mathcal{T}$ the family of open sets; the topology is not necessarily Hausdorff or even $T_0$), $\mathcal{B} \subseteq \mathcal{T}$ a basis (i.e. every member of $\mathcal{T}$ is the union of some subfamily of $\mathcal{B}$), and $|\mathcal{T}| > |\mathcal{B}| + \lambda$, then $|\mathcal{T}| \geq 2^\lambda$.

*Proof:* By [Sh 454a]—the only missing point is that for $\lambda > \aleph_0$, we need: for arbitrarily large $\mu < \lambda$ there is $\kappa \in (\beth_2(\mu)^+, \lambda)$ such that $\mathrm{cov}(|B|, \kappa^+, \kappa^+, \mu) \leq |B|$, which holds by 1.1 (really in the proof there we use 1.4). $\blacksquare_{1.5}$

*1.6 Proof of 1.1:* Assume this fails. By Fodor's lemma (as in 1.3) without loss of generality $\mathrm{cf}(\mu) = \aleph_0$.

Without loss of generality for our given $\mu$, $\lambda$ is the minimal counterexample. Let $\mu = \sum_{n<\omega} \mu_n$, $\mu_n = \mathrm{cf}(\mu_n) < \mu$; so for each $n$ there is $\lambda_n \in (\mu, \lambda)$ such that $\mathrm{pp}_{\Gamma(\mu^+, \mu_n)}(\lambda_n) \geq \lambda$; hence for some $\mathfrak{a}_n \subseteq \mathrm{Reg} \cap (\mu, \lambda_n)$ of cardinality $\leq \mu$ and $\mu_n$-complete ideal $J_n \supseteq J^{\mathrm{bd}}_{\mathfrak{a}_n}$ we have $\lambda_n = \sup(\mathfrak{a}_n)$ and $\Pi\mathfrak{a}_n/J_n$ has true cofinality which is $\geq \lambda$. Let $\theta_n = \mathrm{cf}(\lambda_n)$, so $\mu_n \leq \theta_n \leq |\mathfrak{a}_n|$.

Without loss of generality $\mu_n > \aleph_0$, hence without loss of generality $|\mathfrak{a}_n| < \mu$, hence without loss of generality $|\mathfrak{a}_n| < \mu_{n+1}$ (and really even $|\mathrm{pcf}(\mathfrak{a}_n)| < \mu_{n+1}$), hence the $\theta_n$'s are distinct, hence the $\lambda_n$'s are distinct, and without loss of generality for $n < \omega$ we have $\lambda_n < \lambda_{n+1}$ and $\theta_n < \theta_{n+1} < \mu$, hence necessarily (by $\lambda$'s minimality) $\lambda = \sum_{n<\omega} \lambda_n$, hence without loss of generality (see [Sh:E12, 5.2]) $\mathrm{tcf}\,(\Pi\mathfrak{a}_n, \leq_{J_n}) = \lambda^+$.

It is clear that forcing by a forcing notion $Q$ of cardinality $< \mu$ changes nothing, i.e., we have the same minimal $\lambda$, etc. (only omit some $\mu_n$'s). So without loss of generality $\mu_0 = \theta_0 = |\mathfrak{a}_0| = |\mathrm{pcf}(\mathfrak{a}_0)| = \aleph_1$, and for some increasing sequence $\langle \sigma_i\colon i < \omega_1 \rangle$ of regular cardinals $< \lambda_0$

$(*)$ $\lambda_0 = \sum_{i<\omega_1} \sigma_i$ and $\prod_{i<\omega_1} \sigma_i/\mathcal{D}_{\omega_1}$ has true cofinality $\lambda^+$

($\mathcal{D}_{\omega_1}$ is the club filter on $\omega_1$).

(Of course, we can alternatively use the generalization of normal filters as in [Sh 410, §5], hence avoid forcing.) (How do we force? First by Levy$(\aleph_0, <\mu_0)$ then Levy$(\mu_0, |\mathrm{pcf}(\mathfrak{a}_0)|)$; there is no change in the pcf structure for a set of cardinals $> |\mathrm{pcf}(\mathfrak{a}_0)|$, so now $|\mathfrak{a}_0| = \aleph_1$, $\sup \mathrm{pcf}_{\aleph_1\text{-complete}}(\mathfrak{a}_0) > \lambda$ and $\mathrm{pcf}(\mathfrak{a}_0)$ has cardinality $\aleph_1$; let $\mathfrak{a}_0 = \{\tau_\varepsilon : \varepsilon < \omega_1\}$, $\mathrm{pcf}(\mathfrak{a}_0) = \{\theta_\varepsilon : \varepsilon < \omega_1\}$, choose by induction on $\epsilon < \omega_1$, an ordinal $\zeta(\varepsilon) < \omega_1$ such that $\tau_{\zeta(\varepsilon)} \notin \bigcup\{\mathfrak{b}_{\theta_\xi}[\mathfrak{a}_0] : \xi < \varepsilon \text{ and } \theta_\xi < \lambda\}$, so $\prod_{\varepsilon<\omega_1} \theta_{\zeta(\varepsilon)}/J^{\mathrm{bd}}_{\omega_1}$ is $\lambda^+$-directed, hence wlog$\langle \theta_{\zeta(\varepsilon)} : \varepsilon < \omega_1\rangle$ is strictly increasing so we get $(*)$ and the statement before it.) Without loss of generality

$(*)_1$ $\alpha < \mu_n \Rightarrow |\alpha|^{\aleph_1} + \beth_3(\aleph_1) < \mu_n$ for $n \geq 1$.

Now by [Sh:g, Ch. VI, §1] there is a forcing notion $Q$ of cardinality $\beth_3(\aleph_1)$ $(<\mu!)$ and a name $\underset{\sim}{D}$ of an ultrafilter on the Boolean Algebra $\mathcal{P}(\omega_1)^V$ (i.e. not on subsets of $\omega_1$ which forcing by $Q$ adds) which is normal (for pressing down functions from $V$), extends $\mathcal{D}_{\omega_1}$ and, the main point, the ultrapower $M =: V^{\omega_1}/\underset{\sim}{D}$ (computed in $V^Q$ but the functions are from $V$) satisfies:

$(*)_2$ for every $\kappa > \beth_3(\aleph_1)$ regular or at least $\mathrm{cf}(\kappa) > \beth_3(\aleph_1)$, for some $g_\kappa \in {}^{\omega_1}\mathrm{Ord}$ from $V$ (but depending on the generic subset $G$ of $Q$), the set $\{g/\approx_{\underset{\sim}{D}} : g \in ({}^{\omega_1}\mathrm{Ord})^V, g <_{\underset{\sim}{D}} g_\kappa\}$ is $\kappa$-like (i.e. of cardinality $\kappa$ but every proper initial segment has cardinality $<\kappa$), the order being $<_{\underset{\sim}{D}}$ of course. We shall say in short "$g_\kappa/\underset{\sim}{D}$ is $\kappa$-like"; note that for each $\kappa$ there is at most one such member in $M$ (as the "ordinals" of $M$ are linearly ordered).

However, we should remember $V^{\omega_1}/\underset{\sim}{D}$ is, in general, not well-founded; still there is a canonical elementary embedding $\mathbf{j}$ of $V$ into $M = V^{\omega_1}/\underset{\sim}{D}$ (of course it depends on $G$). Note that $\mathbf{j}$ maps the natural numbers onto $\{x \in M : M \models \text{“}x \in \mathbf{j}(\omega)\text{”}\}$, but this fails for $\omega_1$; without loss of generality $\mathbf{j} \restriction (\omega+1)$ is the identity. If $M \models$ "$x$ an ordinal" let $\mathrm{card}_M(x)$ be the cardinality in $V^Q$ of $\{y : M \models y < x\}$. Note: also $\mathbf{j}(\mu)$ is $\mu$-like and $\{\mathbf{j}(\mu_n) : n < \omega\}$ is unbounded in $\mathbf{j}(\mu)$.

Without loss of generality for every $n \geq 1, \mu_n > |Q|$, and $\mathrm{Min}(\mathfrak{a}_{n+1}) > \lambda_n$. For every regular $\kappa \in (\mu_1, \lambda^+]$ there is $x_\kappa = g_\kappa/\underset{\sim}{D}$ which is $\kappa$-like. Note: $g_\kappa \in V$ (not $\in V^Q \setminus V$), but we need the generic subset of $Q$ to know which member of $V$ it is. Let $\{g_{\kappa,i} : i < i_\kappa\} \in V$ be a set such that $\Vdash_Q$ "for some $i < i_\kappa$ we have $g_{\kappa,i}/\underset{\sim}{D}$ is $\kappa$-like" and $i_\kappa \leq \beth_3(\aleph_1)$. For regular (in $V$) cardinal $\kappa \in (\mu, \lambda^+)$, necessarily $M \models$ "$x_\kappa$ is regular $> \mathbf{j}(\mu)$ and $\leq g_{\lambda^+}/\underset{\sim}{D}$", hence without loss of generality $g_{\lambda^+} = \langle \sigma_\varepsilon : \varepsilon < \omega_1\rangle$ (why? see $(*)$, by [Sh:g, Ch.V] for some normal filter $\mathcal{D}$ on $\omega_1$ and $\sigma'_\varepsilon \le \sigma_\varepsilon$ we have $\prod_{\varepsilon<\omega_1} \sigma'_\varepsilon/\mathcal{D}$ is $\lambda^+$-like, and force as above; by renaming we have the above).

Now also without loss of generality for regular $\kappa \in (\mu, \lambda^+]$ and $i < i_\kappa$ we have $\mathrm{Rang}(g_{\kappa,i})$ is a set of regular cardinals $> \mu$ but $< \lambda_0$ of cardinality $\aleph_1$ (as without loss of generality $g_{\kappa,i}(\varepsilon) < \sigma_\varepsilon$ for $\varepsilon < \omega_1$ and recall $\sigma_\varepsilon < \lambda_0$). For $n \ge 1$ denote $\mathfrak{c}_n =: \bigcup\{\mathrm{Rang}(g_{\kappa,i})\colon \kappa \in \mathfrak{a}_n, i < i_\kappa\}$ and $\mathfrak{d}_n =: \mathbf{j}(\mathfrak{c}_n) \in M$; note $V \models$ "$|\mathfrak{c}_n| \le |\mathfrak{a}_n| + |Q| = |\mathfrak{a}_n|$". So $M \models$ "$\mathfrak{d}_n$ is a set of regular cardinals, each $> \mathbf{j}(\mu)$ but $< \mathbf{j}(\lambda_0)$, of cardinality $\le \mathbf{j}(|\mathfrak{a}_n|) < \mathbf{j}(\mu_{n+1}) < \mathbf{j}(\mu)$". Also for every $\kappa \in \mathfrak{a}_n$ we have $M \models$ "$x_\kappa \in \mathfrak{d}_n$" as $x_\kappa = g_{\kappa,i}/\underset{\sim}{D}$ for some $i < i_\kappa$ and $\mathrm{Rang}(g_{\kappa,i}) \subseteq \mathfrak{c}_n$.

We can apply the theorem on the structure of pcf ([Sh:g, Ch. VIII, 2.6]) in $M$ (as $M$ is elementarily equivalent to $V$) and get $\langle \mathfrak{b}_y[\mathfrak{d}_n]\colon y \in \mathrm{pcf}(\mathfrak{d}_n)\rangle \in M$ and $\langle\langle f^{\mathfrak{d}_n,y}_t\colon t < y\rangle\colon y \in \mathrm{pcf}(\mathfrak{d}_n)\rangle \in M$ (this is not a real sequence, only $M$ "thinks" so).

For $y \in M$ such that $M \models$ "$y$ a limit ordinal (e.g. a cardinal)" let $\lambda_y$ be the cofinality (in $V^Q$) of $(\{x : M \models \text{"}x \text{ an ordinal } < y\text{"}\}, <^M)$. So

$(*)_3$ $\kappa = \lambda_{(x_\kappa)}$ for $\kappa \in \mathrm{Reg}$, $\kappa > |Q|$,

$(*)_4$ assume $|\{a\colon a \in^M \mathbf{j}(\mu_m)\}| < \mu_n$, then $M \models$ "$\sup \mathrm{pcf}_{\mathbf{j}(\mu_m)\text{-complete}}(\mathfrak{d}_n \cap g_{\lambda^+}/\underset{\sim}{D}) \ge g_{\lambda^+}/\underset{\sim}{D}$", assuming for simplicity $1 < m < n$.

[Why? Assume not, so $M \models$ "$\sup \mathrm{pcf}_{\mathbf{j}(\mu_m)\text{-complete}}(\mathfrak{d}_n \cap g_{\lambda^+}/\underset{\sim}{D}) < g_{\lambda^+}/\underset{\sim}{D}$" hence $M \models$ "for every $g \in \Pi(\mathfrak{d}_n \cap g_{\lambda^+}/\underset{\sim}{D})$ for some $\langle (y_\ell, a_\ell)\colon \ell < \mathbf{j}(\mu_m)\rangle$, $y_\ell \in \mathrm{pcf}(\mathfrak{d}_n \cap g_{\lambda^+}/\underset{\sim}{D})$, $a_\ell$ an ordinal $< y_\ell$ we have $g < \sup_{\ell<\mathbf{j}(\mu_m)} f^{y_\ell}_{a_\ell}$". In $V^Q$ we have $\Pi\mathfrak{a}_n/J_n$ is $\lambda^+$-directed, hence $\prod_{\kappa\in\mathfrak{a}_n}(\{t\colon t <^M x_\kappa\}, <^M)/J_n$ is $\lambda^+$-directed (by $(*)_3$), hence there is a function $g^*$ such that

(a) $\mathrm{Dom}(g^*) = \mathfrak{a}_n$,

(b) $g^*(\kappa) <^M x_\kappa = g_\kappa/\underset{\sim}{D}$,

(c) if $M \models$ "$y \in \mathrm{pcf}(\mathfrak{d}_n \cap g_{\lambda^+}/\underset{\sim}{D})$ and $a < y$" then
$\{\kappa \in \mathfrak{a}_n : M \models \text{"}f^{\mathfrak{d}_n,y}_a(x_\kappa) <^M g^*(\kappa)\text{"}\} = \mathfrak{a}_n \bmod J_n$.

By 1.7(1) below we can find $Y \in V$ such that $|Y| < |Q|^+ + \mu = \mu$ and $\kappa \in \mathfrak{a}_n \Rightarrow M \models$ "$g^*(\kappa) \in \mathbf{j}(Y)$". There is $g^\otimes \in M$ such that $M \models$ "$g^\otimes \in \Pi\mathfrak{d}_n$ and $g^\otimes(\theta) = (\sup(\mathbf{j}(Y)) \cap \theta) + 1 < \theta$ for $\Theta \in \mathfrak{d}_n$" (as $M \models$ "$\mathrm{Min}(\mathfrak{d}_n) > \mathbf{j}(\mu)$").

By the choice of $Y$ clearly $\kappa \in \mathfrak{a}_n \Rightarrow g^*(\kappa) <^M g^\otimes(\kappa)$.

By the choice of $\langle\langle f^{\mathfrak{d}_n,y}_t\colon t < y\rangle : y \in \mathrm{pcf}(\mathfrak{d}_n)\rangle$ (in $M$'s sense) and the assumption toward contradiction we have:

$M \models$ "there is a subset $\Theta$ of $\mathrm{pcf}(\mathfrak{d}_n \cap g_{\lambda^+}/\underset{\sim}{D})$ of cardinality $< \mathbf{j}(\mu_m)$ and

$$\langle a_\theta : \theta \in \Theta\rangle \in \Pi\Theta \text{ such that } (\forall \sigma \in \mathfrak{d}_n)(\bigvee_{\theta\in\Theta} g^\otimes(\sigma) < f^{\mathfrak{d}_n,\theta}_{a_\theta}(\sigma))\text{"}.$$

Choose such a sequence $\langle a_\theta : \theta \in \Theta \rangle$ in $M$ and let $\langle \theta_i : i < i(*) \rangle$ list the $\theta \in {}^M\Theta$, so $i(*) < \mu_n$ by the hypothesis of $(*)_4$. Let $\mathfrak{a}_{n,i} = \{\kappa \in \mathfrak{a}_n$: letting $\sigma = x_\kappa \in M$ we have $g^*(\sigma) < f^{\mathfrak{d}_n, \theta_i}_{a_{\theta_i}}(\sigma)\} \in V^Q$. Now as $g^*(\kappa) < g^{\otimes}(x_\kappa)$, clearly $\mathfrak{a}_n = \bigcup_{i<i(*)} \mathfrak{a}_{n,i}$. So for some $i < i(*)$ we have $\mathfrak{a}_{n,i} \in J_n^+$, and we get a contradiction to the choice of $g^*$, hence at last we have proved $(*)_4$.]

Clearly $\mathbf{j}(\langle \mathfrak{c}_n : n < \omega \rangle)$ is a sequence of length $\mathbf{j}(\omega) = \omega$, hence $\mathbf{j}(\langle \mathfrak{c}_n : n < \omega \rangle) = \langle \mathfrak{d}_n : n < \omega \rangle$, i.e. with $n$-th element $\mathfrak{d}_n$. Let $\bar{z} \in M$ be such that $M \models$ "$\mathfrak{p} = \langle (k_n, t_n, s_n) : n < \omega \rangle$ defined by: $k_n < \omega$ is maximal $k$ such that $g_{\lambda^+}/\underset{\sim}{D} \leq \sup \mathrm{pcf}_{\mathbf{j}(\mu_k)\text{-complete}}(\mathfrak{d}_n \cap g_{\lambda^+}/\underset{\sim}{D})$, and $t_n$ is the minimal cardinal $t$ such that $\sup \mathrm{pcf}_{\mathbf{j}(\mu_n)\text{-complete}}(\mathfrak{d}_n \cap t)$ is $\geq g_{\lambda^+}/\underset{\sim}{D}$ and $\mathrm{cf}(t_n) = s_n$ so $s_n \geq \mathbf{j}(\mu_n)$". As $\mathbf{j}(\mu)$ is $\mu$-like clearly $(\forall m < \omega)(\exists n < \omega)(m < n$ and $|\{x \in M : x \in^M (\mathbf{j}(\mu_m))\}| < \mu_n)$ hence by $(*)_4$ above necessarily $(\forall m < \omega)(\exists n < \omega)\,[|[s_n]| \geq \mu_m]$, but $\mathbf{j}(\mu)$ is the limit of $\langle \mathbf{j}(\mu_n) : n < \omega \rangle \in M$, hence $M \models$ "$\mathbf{j}(\mu) = \lim s_n$". Now

$(*)_5$ $M \models$ "$\mathbf{j}(\mu), g_{\lambda^+}/\underset{\sim}{D}$ form a counterexample to the Theorem 1.1".

But as $\mathbf{j}$ is an elementary embedding of $V$ to $M$, the choice of $\lambda$ (minimal) implies

$$M \models \text{" there is no } \lambda' < \mathbf{j}(\lambda) \text{ such that } \mathbf{j}(\mu), \lambda' \text{ form a counterexample to the theorem"}.$$

But as Rang $[g_{\lambda^+}/\underset{\sim}{D}] < \mathbf{j}(\mu_0) < \mathbf{j}(\lambda)$, clearly we have $M \models$ "$g_\lambda/\underset{\sim}{D} < \mathbf{j}(\lambda)$".

By the last two sentences we get a contradiction to $(*)_5$. $\blacksquare_{1.1}$

*1.7 Observation:* Let $Q, \underset{\sim}{D}, G \subseteq Q, V^Q, M, \mathbf{j}$ be as in the proof 1.6. Let for $z \in M$, $[z] = \{t : M \models t \in y\}$. So

(1) If $Y \in V^Q$, $Y \subseteq M$, $\chi = \mathrm{Max}\left\{|Y|^{V^Q}, |Q|^V\right\}$, then for some $y \in V$, $|y|^V \leq \chi$ and $\forall x[x \in Y \Rightarrow M \models$ "$x \in \mathbf{j}(y)$"].

(2) Assume $M \models$ "$\mathfrak{d}$ is a set of regular cardinals $> |\mathfrak{d}|, > \mathbf{j}\left(|Q|^V\right)$" and $\lambda_y$ (when $M \models$ "$y$ limit ordinal") is as in 1.6 (its cofinality in $V^Q$).

   (a) If $M \models$ "$y \in \mathrm{pcf}(\mathfrak{d})$", $J$ is (in $V^Q$) the ideal on $[\mathfrak{d}]$ generated by $\{[\mathfrak{b}_\theta[\mathfrak{d}]] : M \models$ "$\theta \in \mathrm{pcf}(\mathfrak{d})$ and $\theta < y$"$\} \cup \{[\mathfrak{d} \setminus \mathfrak{b}_y[\mathfrak{d}]]\}$, then (in $V^Q$) $\prod_{x \in [\mathfrak{d}]} \lambda_x / J$ has true cofinality $\lambda_y$,

   (b) $\mathrm{cf}\,(\Pi\{\lambda_y : y \in [\mathfrak{d}]\}) = \max\{\lambda_y : y \in [\mathrm{pcf}\ \mathfrak{d}]\}$.

*Proof:* Straightforward (and we use only part (1)). For (2)(b) remember

$$M \models \text{"} y \text{ is finite "} \Rightarrow [y] \text{ finite.}$$

*1.8 Remark:* Of course, the proof of 1.1 gives somewhat more than stated (say after fixing $\mu_0 = \aleph_1$). E.g.,

$\oplus$ the cardinal $\mu$ satisfies the conclusion of 1.1 for $\lambda \geq \lambda^*$ if

$\boxtimes_\mu$ $\mu >$ cf$(\mu) = \aleph_0$ (as before this suffices) and $\mu = \sup\{\kappa < \mu$: $\kappa$ is regular uncountable and there is a forcing notion $Q$ satisfying the $\kappa$-c.c. of cardinality $\leq \lambda_0 < \mu\}$ such that $\Vdash_Q$ "for every $\aleph_1$-complete filter $D$ on $\kappa$ from $V$ containing the co-countable sets there is an ultrafilter $\underset{\sim}{D}$ on $\mathcal{P}(\kappa)^V$ extending $D$ as in [Sh:g, Ch. VI, §1] for regular cardinal $> \lambda^+$ which is complete for partitions of $\kappa$ from $V$ to countably many parts".

Alternatively, we can phrase the theorem after fixing $D$.

## 2. The main theorem revisited

We give another proof and get more refined information. Note that in 2.1 if $\mu$ is strong limit, we can choose $R^*$ such that: if $\theta < \kappa$ are in $R^*$ then $2^\theta < \kappa$, and then $\otimes^0_{R^*,\theta,\theta_1}$ is immediate.

2.1 THEOREM: *Suppose $\mu$ is a limit singular cardinal satisfying:*

$\otimes^0_\mu$ *for any $R \subseteq \mu \cap \mathrm{Reg}$ unbounded, for some $\theta \in R$, $\theta >$ cf$(\mu)$ and $\theta_1$, cf$(\mu) < \theta_1, \aleph_1 \leq \theta_1 < \theta$ and $R^* \subseteq R \setminus \theta^+$ unbounded in $\mu$ we have:*

$\otimes^0_{R^*,\theta,\theta_1}$ *if $\sigma < \kappa$ are in $R^*$, $f_\alpha : \theta \to \sigma$ for $\alpha < \kappa$, $I_\kappa$ a $\kappa$-complete ideal on $\kappa$ extending $J^{\mathrm{bd}}_\kappa$ and $J$ is a $\theta$-complete ideal on $\theta$, <u>then</u> for some $A \in I^+_\kappa$ and $B_\alpha \subseteq \theta$ for $\alpha \in A$ satisfying $B_\alpha = \theta$ mod $J$ we have $\xi < \theta \Rightarrow |\{f_\alpha(\xi) : \alpha \in A$ and $\xi \in B_\alpha\}| < \theta_1$.*

*Then*

$\otimes^1_\mu$ *for every $\lambda > \mu$ we have:*

$\otimes^1_{\lambda,\mu}$ *for some $\kappa < \mu$ we have:*

$\otimes^1_{\lambda,\mu,\kappa}$ *for every $\mathfrak{a} \subseteq (\mu, \lambda) \cap$ Reg of cardinality $< \mu$, $\mathrm{pcf}_{\kappa-\mathrm{complete}}(\mathfrak{a}) \subseteq \lambda$.*

Before we prove it, note:

*2.2 Observation:* Assume:

(a) $\langle w^n_i : i < \alpha^* \rangle$ is a sequence of pairwise disjoint sets, $w^n = \bigcup_{i<\alpha^*} w^n_i$ (possibly $w^n_i = \emptyset$ for some $n$ and $i$),

(b) $\left(\sup\limits_{n,i} |w^n_i|^+\right) < \theta$ and $\theta$ is uncountable,

(c) $J_n$ is a $\theta$-complete ideal on $w^n$ such that $w^n \notin J_n$,

(d) $h_i^n$ or a partial function from $w_i^{n+1}$ to $w_i^n$ and $h^n = \bigcup_{i<\alpha^*} h_i^n$,

(e) for every $A \in J_{n+1}$ the set $\{x \in w^n : (\forall y \in w^{n+1})[h^n(y) = x \Rightarrow y \in A]\}$ belongs to $J_n$.

Then for some $i$ there are $x_n \in w_i^n$ such that $\bigwedge_n h^n(x_{n+1}) = x_n$.

*2.3 Remark:* Hence for the $J_m$-majority of $y \in w^m$ there is $\langle x_n: n < \omega\rangle$ as above such that $y = x_m$.

*Proof:* Without loss of generality $\langle w_i^n: n < \omega, i < \alpha^*\rangle$ are pairwise disjoint. Now we define by induction on the ordinal $\zeta \leq \theta$ for each $i < \alpha^*$ a set $u_i^\zeta \subseteq w_i =: \bigcup_{n<\omega} w_i^n$ by:

$$u_i^\zeta = \left\{x \in w_i: x \in \bigcup_{\xi<\zeta} u_i^\xi \text{ or } (\forall n)(\forall y \in w_i^{n+1})[h_i^n(y) = x \Rightarrow y \in \bigcup_{\xi<\zeta} u_i^\xi]\right\}.$$

So $\langle u_i^\zeta: \zeta < \theta\rangle$ is an increasing sequence of subsets of $w_i$. Also $u_i^{\zeta+1} = u_i^\zeta \Rightarrow (\forall \xi > \zeta)[u_i^\xi = u_i^\zeta]$, hence there is for each $i < \alpha^*$ a unique $\zeta[i] < \aleph_1 + |w_i|^+$ such that $u_i^\zeta = u_i^{\zeta[i]} \Leftrightarrow \zeta \geq \zeta[i]$.

If for some $i$ we have $u_i^{\zeta[i]} \neq w_i$, we can easily prove the conclusion so assume $u_i^{\zeta[i]} = w_i$ for every $i$. Let $\mu = \sup_i(|w_i|^+ + \aleph_1)$, so except when $\theta \leq \aleph_1$ (hence $\theta = \aleph_1 = \mu$) we know $\mu < \theta$. Now we can use clause (e) to prove by induction on $\zeta \leq \mu$ for all $n$ that

$$\bigcup\{u_i^\zeta \cap w_i^n: i < \alpha^*\} \in J_n$$

(we use $J_n$ is $\theta$-complete, $\theta > \mu$). But as $i = \mu \Rightarrow u_i^\mu \cap w_i^n = w_i^n$ we get $w^n \in J_n$, a contradiction. We are left with the case $\theta = \aleph_1$ so each $w_i^n$ is finite and $i < \alpha^* \Rightarrow \zeta[i] < \theta$; but then for each $m$ we have $\bigcup\{u_i^m \cap w_i^0: i < \alpha^*\} \in J_0$, so as $J_0$ is $\theta$-complete there is $x \in w^0$ such that for each $m < \omega$ and $i < \alpha^*$ we have $x \notin u_i^m \cap w_i^0$. For some $i(*)$, $x \in w_{i(*)}^0$, so as $x \notin u_{i(*)}^n$ for some $x_n \in w_{i(*)}^n$ we have $h^{n-1} \circ h^{n-2} \circ \cdots \circ h_0(x_n) = x$. By König's Lemma (as all $w_{i(*)}^n$ are finite) we finish. $\blacksquare_{2.2}$

Before we continue we mention some things which are essentially from [Sh:g] and, more explicitly, [Sh 430, 6.7A].

We forgot there to mention the most obvious demand

2.4 Subclaim: *In* [Sh 430, 6.7A] *we can add:*

(j) $\max \mathrm{pcf}(\mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}]) = \lambda$ *(when defined).*

*Also in* [Sh 430, 6.7] *we can add*

($\delta$) max pcf$(\mathfrak{b}_\lambda) = \lambda$.

*Proof:* This is proved during the proof of [Sh 430, 6.7] (see $(*)_4$ in that proof, p. 103). Actually we have to state it earlier in $(*)_2$ there, i.e., add

($\zeta$) max pcf$(\mathfrak{b}^{i,j}_\lambda) \leq \lambda$.

We then quote [Sh:g, Ch. VIII, 1.3, p. 316], but there this is stated.

Lastly, concerning [Sh 430, 6.7A] the addition is inherited from [Sh 430, 6.7]. $\blacksquare_{2.4}$

2.5 SUBCLAIM: *In* [Sh 430, 6.7A] *we can deduce:*

($\alpha$) *if* $\mathfrak{a}' \subseteq \bigcup_{\beta<\sigma} \mathfrak{a}_\beta$, $|\mathfrak{a}'| < \sigma$, $\mathfrak{a}' \in N_\sigma$, *then for some* $\beta(*) < \sigma$ *and finite* $\mathfrak{c} = \{\theta_0, \theta_1, \ldots, \theta_n\} \subseteq \mathfrak{a}_{\beta(*)}$ *we have*
  - (i) $\theta_\ell > \theta_{\ell+1}$,
  - (ii) $\mathfrak{a}' \subseteq \bigcup_{\ell\leq n} \mathfrak{b}^{\beta(*)}_{\theta_\ell}[\bar{\mathfrak{a}}]$,
  - (iii) $\beta \in (\beta(*), \sigma) \Rightarrow \mathfrak{b}^{\beta}_{\theta_\ell}[\bar{\mathfrak{a}}] \cap \mathfrak{a}' = \mathfrak{b}^{\beta(*)}_{\theta_\ell}[\bar{\mathfrak{a}}] \cap \mathfrak{a}'$,
  - (iv) $\theta_\ell = \max \mathrm{pcf}(\mathfrak{a}' \setminus \bigcup_{k<\ell} \mathfrak{b}^{\beta(*)}_{\theta_k}[\mathfrak{a}])$;

($\beta$) *moreover,* $\langle \theta_\ell \colon \ell \leq n\rangle$ *is definable from* $\mathfrak{a}'$, $\beta(*)$ *and* $\langle \mathfrak{b}^{\beta(*)}_\theta[\bar{\mathfrak{a}}] \colon \theta \in \mathfrak{a}_\beta\rangle$ *uniformly;*

($\gamma$) *if* $\langle \mathfrak{a}'_\varepsilon \colon \varepsilon < \zeta\rangle \in N_\sigma$, $\zeta < \sigma$, $|\mathfrak{a}'_\varepsilon| < \sigma$ *then we can have one* $\beta(*)$ *for all* $\mathfrak{a}'_\varepsilon$ *and so* $\langle\langle \theta_{\varepsilon,\ell} \colon \ell \leq n(\varepsilon)\rangle : \varepsilon < \zeta\rangle \in N_{\beta(*)}$.

*Proof:* Clause ($\alpha$). We choose $\beta_\ell, \theta_\ell$ by induction on $\ell$. For $\ell = 0$ clearly for some $\gamma_0 < \sigma$, $\mathfrak{a}' \in N_{\gamma_0}$ so $\mathfrak{a}' \subseteq \mathfrak{a}_{\gamma_0}$, hence $\theta_0 =$ max pcf$(\mathfrak{a}')$ belongs to $N_{\gamma_0}$ hence to $\mathfrak{a}_\beta$ for $\beta \in [\gamma_0, \sigma)$, so by clause ($i$) of [Sh 430, 6.7A], $\langle \mathfrak{b}^\beta_{\theta_0}[\bar{\mathfrak{a}}] : \beta \in [\gamma_0, \sigma)\rangle$ is increasing hence $\langle \mathfrak{b}^\beta_{\theta_0}[\bar{\mathfrak{a}}] \cap \mathfrak{a}' \colon \beta \in [\gamma_0, \sigma)\rangle$ is eventually constant, say for $\beta \in [\beta_0, \sigma), \beta_0 \in (\gamma_0, \sigma)$. For $\ell + 1$ apply the case $\ell = 0$ to $\mathfrak{a}' \setminus \bigcup_{k\leq\ell} \mathfrak{b}^{\beta_k}_{\theta_k}[\bar{\mathfrak{a}}]$ and get $\theta_{\ell+1}, \beta_{\ell+1}$.

Clauses ($\beta$), ($\gamma$). Easier. $\blacksquare_{2.5}$

2.6 CLAIM: *1) Assume* $\sigma \geq \aleph_0$ *is regular,* $\lambda$ *a cardinal,* $J$ *the* $\sigma$*-complete ideal generated by* $J_{<\lambda}[\mathfrak{a}]$ *for a set* $\mathfrak{a}$ *of regular cardinals* $> |\mathfrak{a}|$, $\mathfrak{a} \notin J$, $\mathfrak{a}_i \in J$ *for* $i < \alpha < |\mathfrak{a}|^+$, $\mathfrak{a} = \bigcup_{i<\alpha} \mathfrak{a}_i$ *and* max pcf$(\mathfrak{a}_i) < \lambda$.

*Then*[3] *we can find* $\mathfrak{b}$, $\mathfrak{b}_i$ $(i < \alpha)$ *and* $I$ *such that:*

(a) $\mathfrak{b}_i \subseteq \mathrm{pcf}(\mathfrak{a}_i)$ *is finite,*

(b) $\mathfrak{b} = \bigcup_{i<\alpha} \mathfrak{b}_i$,

(c) $I$ *is an ideal on* $\mathfrak{b}$ ,

(d) *for* $w \subseteq \alpha$ *we have* $\bigcup_{i\in w} \mathfrak{a}_i \in J \Leftrightarrow \bigcup_{i\in w} \mathfrak{b}_i \in I$,

(e) $I$ *is the* $\sigma$*-complete ideal generated by* $J_{<\lambda}[\mathfrak{b}]$,

3 Note that without loss of generality $i < \alpha \Rightarrow \mathfrak{a}_i \neq \emptyset$, so necessarily $|\alpha| \leq |\mathfrak{a}|$.

(f) *we have* $\mathfrak{b}_i = \{\lambda_{i,\ell}: \ell < n_i\}$ *and if* $I_1$ *is an* $\aleph_1$*-complete ideal on* $\mathfrak{b}$ *extending* $I$ *(so* $I_1 = I$ *is O.K. if* $\sigma > \aleph_0$*), then for any* $\mathfrak{d} \in I_1^+$ *there are* $B \subseteq \alpha$ *and* $\ell^* < \omega$ *such that:*

($\alpha$) $\{\lambda_{i,\ell^*}: i \in B\} \subseteq \mathfrak{d}$,

($\beta$) $\{\lambda_{i,\ell^*}: i \in B\} \in I_1^+$,

($\gamma$) *for every* $B' \subseteq B$ *we have* $\bigcup_{i\in B'} \mathfrak{b}_i \in I_1 \Leftrightarrow \{\lambda_{i,\ell^*}: i \in B'\} \in I_1$.

*2) Assume in addition* $\mathrm{pcf}_{\kappa_i-\mathrm{complete}}(\mathfrak{a}_i) \subseteq \lambda_i$ *and* $\kappa_i \le \sigma$*, then we can find* $\mathfrak{b}$, $\mathfrak{b}_i$ $(i < \alpha)$ *and* $I$ *such that:*

(a)′ $\mathfrak{b}_i \subseteq \mathrm{pcf}_{\kappa_i-\mathrm{complete}}(\mathfrak{a}_i) \subseteq \lambda_i$ *has cardinality* $< \kappa_i$

*and* (b)–(e) *hold.*

*3) Assume*

(i) $I$ *an ideal on* $\alpha$,

(ii) $J$ *an ideal on* $\beta$,

(iii) $\langle \chi_i: i < \alpha\rangle$ *a sequence of regular cardinals with* $\mathrm{tcf}(\prod_{i<\alpha}\chi_i/I) = \chi$,

(iv) *for* $i < \alpha, \langle \tau^i_j: j < \beta\rangle$ *is a sequence of regular cardinals with* $\mathrm{tcf}(\prod_{j<\beta}\tau^i_j/J) = \chi_i$,

(v) $\langle \sigma_j: j < \beta\rangle$ *is a sequence of regular cardinals,*

(vi) $|\alpha| + |\beta| + \sum_{j<\beta}\sigma_j < \min\{\tau^i_j: i < \alpha, j < \beta\}$.

*Then there are for each* $j < \beta$ *an ordinal* $\varepsilon_j < \sigma_j$ *and sets* $\langle \mathfrak{b}^j_\varepsilon: \varepsilon < \varepsilon_j\rangle$ *such that*

(a) $\bigcup_{\varepsilon<\varepsilon_j}\mathfrak{b}^j_\varepsilon \subseteq \{\tau^i_j: i < \alpha\}$ *and if* $\max\ \mathrm{pcf}\{\tau^i_j: i < \alpha, j < \beta\} = \chi$ *then equality holds,*

(b) $\lambda^j_\varepsilon =: \max\ \mathrm{pcf}(\mathfrak{b}^j_\varepsilon)$ *is in* $\mathrm{pcf}_{\sigma_j\text{-complete}}(\mathfrak{b}^j_\varepsilon)$,

(c) *letting* $J^*$ *be the ideal with domain* $\bigcup_{j<\beta}\{j\}\times\varepsilon_j$ *defined by* $A \in J^*$ *iff* $\max\ \mathrm{pcf}\{\lambda^j_\varepsilon: (j,\varepsilon) \in A\} < \chi$*, we have* $\chi = \max\mathrm{pcf}\{\lambda^j_\varepsilon: j < \beta, \varepsilon < \varepsilon_j\}$,

(d) *if* $w \in J^*, \chi$ *then* $\{i < \alpha: \{j < \beta : \exists\varepsilon < \varepsilon_j[\tau^i_j \in \mathfrak{b}^j_\varepsilon \wedge (j,\varepsilon) \in w\} \notin J\} \in I$.

*(Note that* $J^*$ *is a proper ideal and* $\prod_{(j,\varepsilon)\in\ \mathrm{Dom}(J^*)}\lambda^j_\varepsilon/J^*$ *is* $\chi$*-directed by basic pcf theory.)*

*Proof:* By the proof of [Sh:g, Ch. VIII, 1.5] or by [Sh 430, 6.7, 6.7A, 6.7B] (for (1)(f), shrink $A$ to make $n_i$ constantly $n^*$, then prove by induction on $n^*$). In more detail:

1) Without loss of generality $\mathrm{Min}(\mathfrak{a}) > |\mathfrak{a}|^{+3}$. To be able to use [Sh 430] freely in its notation rename $\mathfrak{a}_i$ as $\mathfrak{e}_i$. We apply [Sh 430, 6.7A, p. 104] with $\mathfrak{a}, \kappa, \sigma$ there standing for $\mathfrak{a}, |\mathfrak{a}|^{++}, |\mathfrak{a}|^+$ here and without loss of generality $\langle \mathfrak{e}_i: i < \alpha\rangle \in N_0, \lambda \in N_0$. By Subclaim 2.5 above for each $i < \alpha$ there are $\beta(i) < |\mathfrak{a}|^+$ and finite $\mathfrak{b}_i \subseteq \mathrm{pcf}(\mathfrak{e}_i) \cap \mathfrak{a}_{\beta(i)}$ such that $\beta \in [\beta(i), |\mathfrak{a}|^+) \Rightarrow \mathfrak{e}_i \subseteq \bigcup_{\mu\in\mathfrak{b}_i}\mathfrak{b}^{\beta+1}_\mu[\bar{\mathfrak{a}}]$. Moreover $\langle(\mathfrak{b}_i, \beta(i)): i < \alpha\rangle \in N_{\beta(*)}$ for $\beta(*) = (\sup_{i<\alpha}\beta(i)) + \omega < |\mathfrak{a}|^+$ and let $\mathfrak{b} = \bigcup_{i<\alpha}\mathfrak{b}_i$

and $I = \{\mathfrak{c} \subseteq \mathfrak{b}$: we can find $\zeta < \sigma$ and $\langle \mathfrak{c}_\varepsilon : \varepsilon < \zeta \rangle$ such that $\mathfrak{c} = \bigcup_{\varepsilon<\zeta} \mathfrak{c}_\varepsilon$ and $\max \mathrm{pcf}(\mathfrak{c}_\varepsilon) < \lambda\}$. Let us check all the clauses of the desired conclusion.

Clause (a): $\mathfrak{b}_i \subseteq \mathrm{pcf}(\mathfrak{e}_i)$ is finite.

Holds by the choice of $\mathfrak{b}_i$.

Clause (b): $\mathfrak{b} = \bigcup_{i<\alpha} \mathfrak{b}_i$.

Holds by the choice of $\mathfrak{b}$.

Clause (c): $I$ an ideal on $\mathfrak{b}$. By [Sh:g, Ch. I] and the definition of $I$.

Clause (d): For $w \subseteq \alpha$ we have $\bigcup_{i\in w} \mathfrak{e}_i \in J \Leftrightarrow \bigcup_{i\in w} \mathfrak{b}_i \in I$.

Why? By the definition of $I$ and $J$, it suffices to prove for each subset $w$ of $\alpha$ that

$$\max \mathrm{pcf}(\bigcup_{i\in w} \mathfrak{e}_i) < \lambda \Leftrightarrow \max \mathrm{pcf}(\bigcup_{i\in w} \mathfrak{b}_i) < \lambda.$$

First assume $\max \mathrm{pcf}(\bigcup_{i\in w} \mathfrak{e}_i) < \lambda$. Now $j \in w \Rightarrow \mathfrak{b}_j \subseteq \bigcup_{i\in w} \mathrm{pcf}(\mathfrak{e}_i)$ hence (by [Sh:g, Ch.I, 1.11]) $\mathrm{pcf}(\bigcup_{j\in w} \mathfrak{b}_j) \subseteq \mathrm{pcf}(\bigcup_{i\in w} \mathfrak{e}_i)$ so

$$\max \mathrm{pcf}(\bigcup_{i\in w} \mathfrak{b}_i) \le \max \mathrm{pcf}(\bigcup_{i\in w} \mathfrak{e}_i) < \lambda,$$

as required.

If the other implication fails, then there is $w \subseteq \alpha$ which exemplifies it in $N_{\beta(*)}$ (as all the relevant parameters are in it), so we need only consider $w \in N_{\beta(*)}$. Assuming $w \in N_{\beta(*)}$ and $\max \mathrm{pcf}(\bigcup_{i\in w} \mathfrak{b}_i) < \lambda$ let $\mathfrak{b}' =: \bigcup_{i\in w} \mathfrak{b}_i$, so $\mathfrak{b}' \in N_{\beta(*)} \cap \mathfrak{a}_{\beta(*)}$ and $\max \mathrm{pcf}(\mathfrak{b}') < \lambda$, and by [Sh 430, 6.7A(h)] for some finite $\mathfrak{c} \subseteq \mathrm{pcf}(\mathfrak{b}') \cap N_{\beta(*)}$ we have $\bigcup_{\theta\in\mathfrak{c}} \mathfrak{b}_\theta^{\beta(*)}[\bar{\mathfrak{a}}]$ includes $\mathfrak{b}'$, recalling $\beta(*)$ is a limit ordinal.

By [Sh 430, 6.7A(f)], i.e., smoothness

$$\tau \in \mathfrak{b}' \Rightarrow \mathfrak{b}_\tau^{\beta(*)}[\bar{\mathfrak{a}}] \subseteq \bigcup_{\theta\in\mathfrak{c}} \mathfrak{b}_\theta^{\beta(*)}[\bar{\mathfrak{a}}]$$

hence

$$\begin{aligned} \tau^* \in \bigcup_{i\in w} \mathfrak{e}_i &\Rightarrow \bigvee_{i\in w} \tau^* \in \mathfrak{e}_i \\ &\Rightarrow \bigvee_{i\in w} \tau^* \in \cup\{\mathfrak{b}_\tau^{\beta(*)}[\bar{\mathfrak{a}}]: \tau \in \mathfrak{b}_i\} \Rightarrow \bigvee_{i\in w} \tau^* \in \bigcup_{\theta\in\mathfrak{c}} \mathfrak{b}_\theta^{\beta(*)}[\bar{\mathfrak{a}}] \\ &\Rightarrow \tau^* \in \bigcup_{\theta\in\mathfrak{c}} \mathfrak{b}_\theta^{\beta(*)}[\bar{\mathfrak{a}}]. \end{aligned}$$

So $\bigcup_{i\in w} \mathfrak{e}_i \subseteq \bigcup_{\theta\in\mathfrak{c}} \mathfrak{b}_\theta^{\beta(*)}[\bar{\mathfrak{a}}]$ hence

$$\max \mathrm{pcf}(\bigcup_{i\in w} \mathfrak{e}_i) \leq \max \mathrm{pcf}(\bigcup_{\theta\in\mathfrak{c}} \mathfrak{b}_\theta^{\beta(*)}[\bar{\mathfrak{a}}])$$
$$\leq \max_{\theta\in\mathfrak{c}}(\max \mathrm{pcf}\ \mathfrak{b}_\theta^{\beta(*)}[\mathfrak{a}]) = \max(\mathfrak{c}) < \lambda$$

(we use Subclaim 2.5 above).

Clause (e): $I$ is the $\sigma$-complete ideal generated by $J_{<\lambda}[\mathfrak{b}]$.

By the choice of $I$.

Clause (f): As $I_1$ is $\aleph_1$-complete for some $n^*$ the set $\mathfrak{d}\cap\bigcup\{\mathfrak{b}_i : |\mathfrak{b}_i| = n^*\}$ belongs to $I_1^+$. Now we try to choose by induction on $\ell \leq n^*+1$ a set $B_\ell \subseteq \alpha$ decreasing with $\ell$ such that:

($\alpha$) $\{\lambda_{i,k} \in \mathfrak{d}\colon i \in B_\ell \text{ and } k \leq \ell\} \in I_1^+$,

($\beta$) for each $k < \ell$ the set $\{\lambda_{i,k}\colon i \in B_\ell\}$ belongs to $I_1$.

For $\ell = 0$, the set $B_0 = \{i < \alpha\colon |\mathfrak{b}_i| = n^*\}$ is O.K.: in clause ($\alpha$) we ask $\bigcup_{i<\alpha} \mathfrak{b}_i \cap \mathfrak{d} \in I_1^+$, by which we mean $\mathfrak{d} \in I_1^+$ which is assumed, and Clause ($\beta$) is empty (no $k < \ell$!); lastly by the choice of $n^*$ we are done.

For $\ell+1$, if $\ell, B_\ell$ are not as required, then there is $B' \subseteq B_\ell$ such that the statements

$$\text{“}\bigcup_{i\in B'} \mathfrak{b}_i \in I_1\text{” and “}\{\lambda_{i,\ell}\colon i \in B'\} \in I_1\text{” have different truth values.}$$

By obvious monotonicity this means $\bigcup_{i\in B'} \mathfrak{b}_i \notin I_1, \{\lambda_{i,\ell}\colon i \in B'\} \in I_1$ so let $B_{\ell+1} = B'$.

If $B_{n^*+1}$ is well defined we have by clause ($\alpha$) that $\{\lambda_{i,k} : i \in B_{n^*+1} \text{ and } k \geq n^*+1\} \in I_1^+$ but as $B_{n^*+1} \subseteq B_0$ this set is empty, easy contradiction.

2) Same proof except that, for defining $\mathfrak{b}_i$, instead of quoting 2.5 we use [Sh 430, 6.7A(h)$^+$]. We could have used it in the proof of part (1) here.

3) We apply [Sh 430, 6.7A] to $\mathfrak{a} =: \{\tau_j^i\colon i < \alpha, j < \beta\} \cup \{\chi_i\colon i < \alpha\}$ and without loss of generality $\langle \chi_i\colon i < \alpha\rangle, I, J, \langle\sigma_j\colon j < \beta\rangle$ and $\langle\langle\tau_j^i\colon j < \beta\rangle : i < \alpha\rangle$ belong to $N_0$. Let $\mathfrak{a}^* \in J_{\leq\chi}[\mathfrak{a}]$ be such that $J_{\leq\chi}[\mathfrak{a}] = J_{<\chi}[\mathfrak{a}] + \mathfrak{a}^*$ and let $\mathfrak{e}_j = \{\tau_j^i\colon i < \alpha\} \cap \mathfrak{a}^*$ but if possible $\mathfrak{a}^* = \mathfrak{a}$. Again by [Sh 430, 6.7A(h)$^+$] for each $j$ there is $\mathfrak{c}_j \subseteq \mathrm{pcf}_{\sigma_j\text{-complete}}(\mathfrak{e}_j)$ such that $\mathfrak{e}_j \subseteq \bigcup_{\theta\in\mathfrak{c}_j} \mathfrak{b}_\theta^{\beta+1}[\bar{\mathfrak{a}}]$. Let $\mathfrak{c}_j = \{\lambda_\varepsilon^j\colon \varepsilon < \varepsilon_j\}$ with no repetitions and let $\mathfrak{b}_\varepsilon^j = \mathfrak{b}_{\lambda_\varepsilon^j}^{\beta+1}[\bar{\mathfrak{a}}] \cap \mathfrak{e}_j$.

Now clause (a) holds by the choices of $\mathfrak{c}_j$ and $\mathfrak{b}_\varepsilon^j$. As for clause (b), note $\max \mathrm{pcf}(\mathfrak{b}_\varepsilon^j) = \lambda_\varepsilon^j$ by 2.4, i.e., clause (j) of [Sh 430, 6.7A] and clearly $\lambda_\varepsilon^j \in \mathrm{pcf}_{\sigma_j\text{-complete}}(\mathfrak{e}_j)$, but $\lambda_\varepsilon^j \notin \mathrm{pcf}(\mathfrak{e}_j \setminus \mathfrak{b}_\varepsilon^j)$ by clause (e) of [Sh 430, 6.7A] so necessarily $\lambda_\varepsilon^j \in \mathrm{pcf}_{\sigma_j\text{-complete}}(\mathfrak{b}_\varepsilon^j)$, that is clause (b) holds.

Let $J^*$ be the ideal with domain $\bigcup_{j<\beta}\{j\} \times \varepsilon_j$ defined by

$$J^* = \{A \subseteq \text{ Dom}(J^*)\colon \max \text{pcf}\{\lambda^j_\varepsilon\colon (j,\varepsilon) \in A\} < \chi\}.$$

By transitivity of pcf, $\chi \in \text{ pcf}(\{\tau^i_j\colon i < \alpha, j < \beta\})$ hence by the choice of $\mathfrak{a}^*, \mathfrak{e}_j$ clearly $\chi = \max \text{pcf}(\bigcup_{j<\beta} \mathfrak{e}_j)$.

As in the proof of clause (d) of part (1) we have

(∗) for $w \subseteq \alpha$ we have

$$\max \text{pcf}(\bigcup_{i\in w} \mathfrak{e}_i) < \chi \Leftrightarrow \max \text{pcf}(\bigcup_{i\in w} \mathfrak{c}_i) < \chi.$$

We conclude that $\chi = \max \text{pcf}(\bigcup_{\ell<\alpha} \mathfrak{c}_i)$, hence $J^*$ satisfies clause (c) (well maybe $\mathfrak{c}_{i_1} \cap \mathfrak{c}_{i_2} \neq \emptyset$? Remember [Sh:g, Ch. I, §1]).

Lastly, we prove clause (d) so assume $w \in J^*$; so by the definition of $J^*$, we have $\max \text{pcf}(\mathfrak{d}) < \chi$ where $\mathfrak{d} = \{\lambda^j_\varepsilon\colon (j,\varepsilon) \in w\}$. So by transitivity of pcf ([Sh:g, Ch. I, 1.11]) as $\chi = \text{ tcf}(\prod_{i<\alpha} \chi_i/I)$ necessarily $B =: \{i < \alpha\colon \chi_i \in \text{ pcf}(\mathfrak{d})\} \in I$. Now for each $i \in \alpha \setminus B$ we have $\chi_i \notin \text{ pcf}(\mathfrak{d})$ hence $\chi_i \notin \text{ pcf}(\mathfrak{d} \cap \mathfrak{e}_i)$; but $\prod_{j<\beta} \tau^i_j/J$ has true cofinality $\chi_i$, so necessarily $B_i =: \{j < \beta\colon \tau^i_j \in \mathfrak{d} \cap \mathfrak{e}_i\} \in J$. Checking the meaning you get clause (d). $\blacksquare_{2.6}$

*2.7 Observation:* If $\kappa > \aleph_0$, $\lambda \in \text{ pcf}_{\kappa\text{-complete}}(\mathfrak{a})$, then for some $\theta$, $\kappa \leq \theta = \text{cf}(\theta) \leq |\mathfrak{a}|$, and $\langle \chi_i\colon i < \theta\rangle$ we have: $\chi_i$ regular, $\chi_i \in \lambda \cap \text{ pcf}(\mathfrak{a})$ and for some $\theta$-complete ideal $I \supseteq J^{\text{bd}}_\theta$ we have $\lambda = \text{ tcf}(\prod_{i<\theta} \chi_i/I)$.

*Proof:* Without loss of generality $\lambda = \max \text{pcf}(\mathfrak{a})$, otherwise replace it by $\mathfrak{b}_\lambda[\mathfrak{a}]$; let $J$ be the $\kappa$-complete filter on $\mathfrak{a}$ which $J_{<\lambda}[\mathfrak{a}]$ generates. Let $\theta$ be minimal such that $J$ is not $\theta^+$-complete so necessarily $\kappa \leq \theta = \text{ cf}(\theta) \leq |\mathfrak{a}|$; as we can replace $\mathfrak{a}$ by any $\mathfrak{a}' \subseteq \mathfrak{a}, \mathfrak{a}' \notin J_{<\lambda}[\mathfrak{a}]$ without loss of generality $\mathfrak{a}$ is the union of $\theta$ members of $J$, so for some $\mathfrak{a}_i \in J$ (for $i < \theta$) we have $\mathfrak{a} = \bigcup_{i<\theta} \mathfrak{a}_i$; as $J$ is $\theta$-complete without loss of generality $\mathfrak{a}_i \in J_{<\lambda}[\mathfrak{a}]$. By 2.6(1), we have $\langle \mathfrak{b}_i\colon i < \theta\rangle, \mathfrak{b}$ and $I$ as there. As $J$ is $\theta$-complete $\{\bigcup_{i\in w} \mathfrak{b}_i\colon |w| < \theta\} \subseteq I$, so by applying clause (f), we can finish. $\blacksquare_{2.7}$

*Proof of 2.1:* We shall prove $\otimes^1_{\lambda,\mu}$ by induction on $\lambda$. Arriving at $\lambda$, assume it is a counterexample so necessarily $\lambda > \mu$, $\text{cf}(\lambda) = \text{ cf}(\mu)$. For each $\kappa < \mu$ there is $\mathfrak{a} \subseteq (\mu, \lambda)$ such that $|\mathfrak{a}| < \mu$ and $\text{pcf}_{\kappa\text{-complete}}(\mathfrak{a}) \nsubseteq \lambda$, so by [Sh:g, ChIX, 4.1] without loss of generality for some $\kappa$-complete ideal $J$ on $\mathfrak{a}$, $\lambda^+ = \text{ tcf}(\Pi\mathfrak{a}/J)$. So (by 2.7) the following subset of $(\mathrm{cf}(\mu), \mu) \cap$ Reg is unbounded in $\mu$ (by 2.7):

$$R =: \Big\{ \theta\colon \mathrm{cf}(\mu) < \theta = \mathrm{cf}(\theta) < \mu \text{ and there is } \langle \chi_{\theta,\zeta} : \zeta < \theta \rangle, \\ \text{a sequence of regular cardinals } \in (\mu, \lambda) \\ \text{and a } \theta\text{-complete ideal } I_\theta \text{ on } \theta \text{ extending } J^{\mathrm{bd}}_\theta \text{ such that} \\ \prod_{\zeta<\theta} \chi_{\theta,\zeta}/I_\theta \text{ has true cofinality } \lambda^+ \Big\}.$$

Let $\theta, \theta_1, R^*$ be witnesses for $\otimes^0_\mu$ (i.e. $\otimes^0_{R^*,\theta,\theta_1}$ holds); without loss of generality $\mathrm{otp}(R^*) = \mathrm{cf}(\mu)$ and remember $\mathrm{cf}(\mu) < \theta_1$, $\theta^+ < \mathrm{Min}(R^*)$, $\theta \in R$. Let $\alpha^* = \theta$; we now define by induction on $n$ the following: $J_n$, $w^n$, $\langle w^n_i \colon i < \theta \rangle$, $\langle \lambda_x \colon x \in w^n \rangle$, $h^n$ as in Observation 2.2 such that $\{x \in w^n : \lambda_x \leq \mu^+\} \in J_n$ and $h^n_i(y) = x \Rightarrow \lambda_y < \lambda_x$, so we shall get a contradiction (the domain of $h_n$ is $\{x \in w^{n+1} \colon \lambda_x > \mu^+\}$). We also demand $\prod_{x\in w^n} \lambda_x/J_n$ is $\lambda^+$-directed and $[x \in w^n \Rightarrow \mu^+ \leq \lambda_x < \lambda]$ hence $\lambda^+_x < \lambda$. We let $w^0_i = \{i\}, \lambda_i = \chi_{\theta,i}$, and $J_0 = I_\theta$. Suppose all have been defined for $n$. Now by the induction hypothesis on $\lambda$ (as $\mu = \sup(R^*)$) for every $x \in w_n$, if $\lambda_x > \mu^+$ then for some $\sigma = \sigma[\lambda_x] \in R^*$ we have

$$\mathfrak{a} \subseteq (\mu, \lambda_x) \ \&\ |\mathfrak{a}| < \mu \Rightarrow \mathrm{pcf}_{\sigma\text{-complete}}(\mathfrak{a}) \subseteq \lambda_x.$$

Remember $J_n$ is $|R^*|^+$-complete (as $\theta > \mathrm{cf}(\mu)$), so it is enough to deal separately with each $u^{n,\sigma} = u(n,\sigma) =: \{x \in w^n \colon \sigma[\lambda_x] = \sigma \text{ and } \lambda_x > \mu^+\}$ where $\sigma \in R^*$. If $u^{n,\sigma} \in J_n$ we have nothing to do. Otherwise choose $\kappa_\sigma \in R^*$, $\kappa_\sigma > \sigma, \theta$ and $I_{\kappa_\sigma}$, $\langle \chi_{\kappa_\sigma,\zeta} \colon \zeta < \kappa_\sigma \rangle$ witnessing $\kappa_\sigma \in R$. By [Sh:g, Ch. IX, 4.1] applied to $\chi_{\kappa_\sigma,\zeta} < \lambda^+ = \mathrm{tcf} \prod_{x\in u(n,\sigma)} \lambda_x/J_n$, for each $\zeta < \kappa_\sigma$ we can find a sequence $\langle \tau^{n,\sigma,\zeta}_x \colon x \in u^{n,\sigma} \rangle$, $\tau^{n,\sigma,\zeta}_x$ regular $< \lambda_x$ but $\geq \mu^+$ and $\prod_{x\in u(n,\sigma)} \tau^{n,\sigma,\zeta}_x/J_n$ has true cofinality $\chi_{\kappa_\sigma,\zeta}$.

Now apply 2.6(3) with $\alpha, \beta, I, J, \chi, \langle \chi_i \colon i < \alpha \rangle, \langle \tau^i_j \colon j < \beta \rangle, \langle \sigma_j \colon j < \beta \rangle$ there standing for $\kappa_\sigma, u(n,\sigma), I_{\kappa_\sigma}, J_n \restriction u(n,\sigma), \lambda^+, \langle \chi_{\kappa_\sigma,\zeta} \colon \zeta < \kappa_\sigma \rangle, \langle \tau^{n,\sigma,\zeta}_x \colon x \in u(n,\sigma) \rangle, \langle \sigma \colon x \in u(n,\sigma) \rangle$. This gives us objects $\langle \mathfrak{b}^{n,\sigma,\varepsilon}_x \colon x \in u(n,\sigma), \varepsilon < \varepsilon_x \rangle$ and $J^{n,\sigma}$ as there. We could have changed some values of $\tau^{n,\sigma,\zeta}_x$ to $\mu^+$ to guarantee that $\lambda^+ = \max \mathrm{pcf}\{\tau^{n,\sigma,\zeta}_x \colon x \in u(n,\sigma), \zeta < \kappa_\sigma\}$, so without loss of generality

$$\{\tau^{n,\sigma,\zeta}_x \colon \zeta < \kappa_\sigma\} = \bigcup_{\varepsilon < \varepsilon_x} \mathfrak{b}^{n,\sigma,\varepsilon}_x.$$

By 2.6(3) clause (d), we have

$(*)_1$ if $w \subseteq \mathrm{Dom}(J^{n,\sigma})$ and

$$\{\zeta < \kappa_\sigma \colon \{x \in u(n,\sigma) \colon (\exists \varepsilon < \varepsilon_x)[\tau^{n,\sigma,\zeta}_x \in \mathfrak{b}^{n,\sigma,\varepsilon}_x \& (x,\varepsilon) \in w]\} \notin J_n\} \notin I_{\kappa_\sigma},$$

then $w \notin J^{n,\sigma}$.

Let $I^{n,\sigma}$ be the ideal on $\mathrm{Dom}(J^{n,\sigma})$ defined by

$$w \in I^{n,\sigma} \Leftrightarrow \big\{\zeta < \kappa_\sigma \colon \{x \in u(n,\sigma) \colon (\exists \varepsilon < \varepsilon_x)[\tau_x^{n,\sigma,\varepsilon} \in \mathfrak{b}_x^{n,\sigma,\varepsilon} \ \& \ (x,\varepsilon) \in w]\} \notin J_n\big\} \in I_{\kappa_\sigma}.$$

Now $(*)_1$ tells us that $J^{n,\sigma} \subseteq I^{n,\sigma}$. Note that since $I_{\kappa_\sigma}$ and $J_n$ are $\theta$-complete proper ideals—we assumed $u(n,\sigma) \notin J_n$—we have that $I^{n,\sigma}$ is a $\theta$-complete proper ideal on $\mathrm{Dom}(J^{n,\sigma})$. This means that if we want to verify that a set is not in the $\theta$-complete ideal generated by $J^{n,\sigma}$, it suffices to see it is not in $I^{n,\sigma}$.

By 2.6(3), (b) we have $\lambda_x^{n,\sigma,\varepsilon} =\colon$ max $\mathrm{pcf}(\mathfrak{b}_x^{n,\sigma,\varepsilon})$ is in $\mathrm{pcf}_{\sigma\text{-complete}}(\mathfrak{b}_x^{n,\sigma,\varepsilon})$. Since $\mathfrak{b}_x^{n,\sigma,\varepsilon} \subseteq \lambda_x$, our choice of $\sigma[\lambda_x] = \sigma$ guarantees

$(*)_2$ $\lambda_x^{n,\sigma,\varepsilon} =$ max $\mathrm{pcf}(\mathfrak{b}_x^{n,\sigma,\varepsilon}) < \lambda_x$.

For $\zeta < \kappa_\sigma$, let $f_\zeta^{n,\sigma}\colon u(n,\sigma) \to \sigma$ be defined by $f_\zeta(x) =$ Min$\{\epsilon < \epsilon_x \colon \tau_x^{n,\sigma,\zeta} \in \mathfrak{b}_x^{n,\sigma,\epsilon}\}$. Now we can apply the choice of $\theta_1, \theta$ (i.e., for them $\otimes^0_{R^*,\theta,\theta_1}$ holds), only instead of "$J$ a $\theta$-complete ideal on $\theta$" we have here "$J_n$ is a $\theta$-complete ideal on a set of cardinality $\theta$ and actually use $J_n \restriction u^{n,\sigma}$". So we get $A^{n,\sigma} \in I^+_{\kappa_\sigma}$ and $B_\zeta^{n,\sigma} = u(n,\sigma)$ mod $J_n$ for $\zeta \in A^{n,\sigma}$ such that:

$$(*)_3 \qquad x \in u^{n,\sigma} \Rightarrow \theta_1 > |\{f_\zeta^{n,\sigma}(x) \colon \zeta \in A^{n,\sigma},\ x \in B_\zeta^{n,\sigma}\}|.$$

Let us define

$$w_{i,\sigma}^{n+1} = \{(x,\sigma,\epsilon) \colon (\exists \zeta \in A^{n,\sigma})[x \in B_\zeta^{n,\sigma} \text{ and } \epsilon = f_\zeta^{n,\sigma}(x) \text{ and } x \in w_i^n]\},$$
$$h_{i,\sigma}^n : w_{i,\sigma}^{n+1} \to w_i^n \text{ is } h_{i,\sigma}^n((x,\sigma,\epsilon)) = x \text{ when } \lambda_x > \mu^+,$$
$$x \in u^{n,\sigma} \Rightarrow \lambda_{(x,\sigma,\epsilon)} = \lambda_x^{n,\sigma,\varepsilon}.$$

Recall we are assuming $u^{n,\sigma} \in J_n^+$; if $i \in u^{n,\sigma} \in J_n$ we let $w_{i,\sigma}^{n+1} = \emptyset$. Now we switch "integrating" on all $\sigma \in R^*$:

$$w_i^{n+1} = \bigcup_{\sigma \in R^*} w_{i,\sigma}^{n+1}.$$

We let

$$w^{n+1} = \bigcup_{\sigma \in R^*} \bigcup_{i<\theta} w_{i,\sigma}^{n+1}, h^n = \bigcup_{\sigma \in R^*} \bigcup_{i<\theta} h_{i,\sigma}^n;$$

$$J_{n+1} = \Big\{ u \subseteq w^{n+1} \colon \text{ for some } i < \theta \text{ and } u_j \subseteq u \text{ for } j < i \text{ we have } u = \bigcup_{i<j} u_j \text{ and for each } j < i \text{ we have } \lambda^+ > \text{ max pcf}\{\lambda_{(x,\sigma,\epsilon)} \colon (x,\sigma,\epsilon) \in u_j\} \Big\}.$$

Most of the verification that $w^{n+1}, h^n$ and $J_{n+1}$ are as required is routine; we concentrate on a few important points

$\boxtimes_0$ $|w_i^{n+1}| < \theta_1$.
[Why? By $(*)_3$, as $\mathrm{cf}(\mu) < \theta_1 < \theta$ so the $\varepsilon$ do not cause a problem.]

$\boxtimes_1$ if $x \in w^n, \lambda_x > \mu^+$ and $h^n(y) = x$, then $\lambda_y < \lambda_x$.
[Why? Choose $\sigma$ such that $x \in u(n,\sigma)$. If $u(n,\sigma) \in J_n$ then $\lambda_y = \mu^+ < \lambda_x$. If $u(n,\sigma) \notin J_n$ then we are done by $(*)_2$.]

$\boxtimes_2$ $w^{n+1} \notin J_{n+1}$.
[Why? Choose $\sigma \in R^*$ with $u(n,\sigma) \notin J_n$, and let $v(n,\sigma) = \{(x,\varepsilon)\colon (x,\sigma,\varepsilon) \in w_\sigma^{n+1}\}$.
For $\zeta \in A^{n,\sigma}$,

$$B_\zeta^{n,\sigma} \subseteq \{x \in u(n,\sigma) : (\exists \varepsilon < \varepsilon_x)[\tau_x^{n,\sigma,\zeta} \in \mathfrak{b}_x^{n,\sigma,\varepsilon} \wedge (x,\varepsilon) \in v(n,\sigma)]\},$$

and so $v(n,\sigma) \notin I^{n,\sigma}$. Thus $v(n,\sigma)$ is not in the $\theta$-complete ideal generated by $J^{n,\sigma}$, and the definitions of $J^{n,\sigma}$ and $J_{n+1}$ imply $w_\sigma^{n+1} \notin J_{n+1}$.]

$\boxtimes_3$ For every $A \in J_{n+1}, B =: \{x \in w^n\colon (\forall y \in w^{n+1})[h^n(y) = x \Rightarrow y \in A]\}$ belongs to $J_n$. [Why? Suppose toward a contradiction that $B \in J_n^+$, and choose $\sigma \in R^*$ such that $B \cap u(n,\sigma) \in J_n^+$. Let $A_1 = \{(x,\sigma,\varepsilon) \in w^{n+1}\colon x \in B\}$, and let $A' = \{(x,\varepsilon)\colon (x,\sigma,\varepsilon) \in A\}$. For $\zeta \in A^{n,\sigma}$ as $B_\zeta^{n,\sigma} = u(n,\sigma) \bmod J_n$ clearly $B \cap B_\zeta^{n,\sigma} \in J_n^+$; also

$$B \cap B_\zeta^{n,\sigma} \subseteq \{x \in u(n,\sigma)\colon (\exists \varepsilon < \varepsilon_x)[\tau_x^{n,\sigma,\zeta} \in \mathfrak{b}_x^{n,\sigma,\varepsilon} \wedge (x,\varepsilon) \in A']\},$$

and since $B \cap B_\zeta^{n,\sigma} \in J_n^+$, by the definition of $I^{n,\sigma}$ we know $A' \notin I^{n,\sigma}$ hence $A_1 \notin J_{n+1}$ but by the definition of $B, A$, clearly $A_1 \subseteq A$, hence $A \notin J_{n+1}$, contradiction.]

Thus we have carried out the induction and hence get by 2.2 the contradiction and finish the proof. $\blacksquare_{2.1}$

2.8 *Remark:* 1) We can be more specific phrasing 2.1: let $R^* \subseteq \mu$ be unbounded, $\bar{\Gamma} = \langle \Gamma_\sigma\colon \sigma \in R^* \rangle$, $\Gamma_\sigma$ a set of ideals on $\sigma$; the desired conclusion is: for every $\lambda > \mu$ for some $\sigma^* < \mu$ we have: if $\sigma \in R^* \setminus \sigma^*$, $\lambda_i \in (\mu,\lambda) \cap$ Reg for $i < \sigma$, $J$, $J \in \Gamma_\sigma$ then $\mathrm{pcf}_{\Gamma_\sigma}\left(\prod_{i<\sigma} \lambda_i, \leq_J\right) \subseteq \lambda$. (Reg is the class of regular cardinals.)
2) You can read the proofs for the case $\mu$ strong limit singular and get an alternative to the proof in §1.

2.9 Claim: *Assume $\lambda^* > \mu > \aleph_1, \mu$ an uncountable limit cardinal and we have:*
$\otimes^{1.5}_{\lambda^*,\mu}$ *for every $\lambda \in (\mu, \lambda^*]$, we have $\otimes^1_{\lambda,\mu}$ (from the conclusion of 2.1).*
*Then*

$\otimes^2_{\lambda^*,\mu}$ ($\alpha$) $\mathfrak{a} \subseteq (\mu, \lambda^*), \mathfrak{a} \subseteq \mathrm{Reg}, |\mathfrak{a}| < \mu \Rightarrow |\lambda^* \cap \mathrm{pcf}(\mathfrak{a})| \leq \mu$,
($\beta$) *if $\mu$ is regular then (for $\mathfrak{a} \subseteq \mathrm{Reg}$):*

$$\mathfrak{a} \subseteq (\mu, \lambda^*) \ \& \ |\mathfrak{a}| < \mu \Rightarrow |\lambda^* \cap \mathrm{pcf}(\mathfrak{a})| < \mu.$$

*Proof:* Let $\mu(*) = \begin{cases} \mu^+ & \text{if } \mu \text{ is singular,} \\ \mu & \text{if } \mu \text{ is regular.} \end{cases}$

So assume $\mathfrak{a} \subseteq (\mu, \lambda^*) \cap \mathrm{Reg}, |\mathfrak{a}| < \mu$, and $\lambda^* \cap$ pcf($\mathfrak{a}$) has cardinality $\geq \mu(*)$. Let $\lambda_0 =$ Min($\mathfrak{a}$) and $\langle \lambda_{i+1} \colon i \leq \mu(*) \rangle$ list the first $(\mu(*) + 1)$-members of $(\mathrm{pcf}(\mathfrak{a})) \setminus \{\lambda_0\}$ (remember pcf($\mathfrak{a}$) has a last member), and for limit $\delta \leq \mu(*)$, let $\lambda_\delta = \bigcup_{i<\delta} \lambda_i$ so $\lambda_{\mu(*)} \leq \lambda^*$. Now by an assumption for some $\kappa < \mu$, $\otimes^1_{\lambda_{\mu(*)},\mu,\kappa}$ (from 2.1), without loss of generality $\kappa$ is regular. Now choose by induction on $\zeta < \mu$, $i(\zeta)$ such that $i(\zeta) < \mu(*)$ is a successor ordinal, $i(\zeta) > \bigcup_{\xi<\zeta} i(\xi)$, and $\lambda_{i(\zeta)} > \sup \mathrm{pcf}_{\kappa\text{-complete}}(\{\lambda_{i(\xi)} \colon \xi < \zeta\})$.

Why is this possible? We know $\mathrm{pcf}_{\kappa\text{-complete}}(\{\lambda_{i(\xi)} \colon \xi < \zeta\})$ cannot have a member $\geq \lambda_{\mu(*)}$ (hence $> \lambda_{\mu(*)}$ being regular), by the choice of $\kappa$. Also $\mathrm{pcf}_{\kappa\text{-complete}}(\{\lambda_{i(\xi)} \colon \xi < \zeta\})$ cannot be unbounded in $\lambda_{\mu(*)}$ (because $\mathrm{cf}(\lambda_{\mu(*)}) = \mu(*) \geq \kappa$ (remember $\mu(*)$ is regular) as then it will have a member $> \lambda_{\mu(*)}$; see [Sh:g, Ch.I, 1.11]). So it is bounded below $\lambda_{\mu(*)}$, hence $i(\zeta)$ exists.

Now we get a contradiction to [Sh 410, 3.5], version (b) of (iv) there (use, e.g., $\langle \lambda_{i(\zeta)} : \zeta < (\kappa + |\mathfrak{a}|)^{+4} \rangle$) (alternatively to [Sh 430, 6.7F(5)]). $\blacksquare_{2.9}$

2.10 THEOREM: *Let $\mu$ be a limit uncountable singular cardinal, $\mu < \lambda$ and* $[|\mathfrak{a}| < \mu$ and $\mathfrak{a} \subseteq \mathrm{Reg} \cap (\mu, \lambda) \Rightarrow |\lambda \cap \mathrm{pcf}(\mathfrak{a})| < \mu]$, *or at least:*

$\oplus_{\mu,\lambda}$ *for every large enough $\sigma \in \mathrm{Reg} \cap \mu$, we have:*

$\oplus^\sigma_{\mu,\lambda}$ *if $\mathfrak{a} \subseteq \mathrm{Reg} \cap (\mu, \lambda), |\mathfrak{a}| < \mu$, then $|\lambda \cap \mathrm{pcf}_{\sigma\text{-complete}}(\mathfrak{a})| < \mu$.*

*Then for every large enough $\kappa < \mu$ we have $\otimes^1_{\mu,\kappa}$ of 2.1, hence* $\mathrm{cov}(\lambda, \mu, \mu, \kappa) = \lambda$.

*Remark:* This proof relies on [Sh 420, §5].

*Proof:* Without loss of generality $\mathrm{cf}(\mu) = \aleph_0$ (e.g., force by Levy($\aleph_0$,cf($\mu$)) as nothing relevant is changed by the forcing, or argue as in 1.3 as $\oplus^\sigma_{\mu,\lambda}$ implies that, for each $\chi \in [\mathrm{cf}\mu, \mu]$, the cardinal $\sup\{|\lambda \cap \mathrm{pcf}_{\sigma\text{-complete}}(\mathfrak{a})| : \mathfrak{a} \subseteq \mathrm{Reg} \cap (\mu, \lambda), |\mathfrak{a}| \leq \chi\}$ is $< \mu$; however, we can just repeat the proof).

Assume this fails. Without loss of generality $\lambda$ is minimal, so $\mathrm{cf}(\lambda) = \aleph_0$. Failure means (by 2.7) that $\mu = \sup(R)$ when

$$R = \Big\{\theta\colon \theta \in \mu \cap \text{ Reg and for some } \chi_\zeta \in \text{ Reg } \cap (\mu, \lambda) \text{ for } \zeta < \theta, \\ \text{and } \theta\text{-complete ideal } I \text{ on } \theta \\ \text{we have } \lambda^+ = \mathrm{tcf}(\prod_{\zeta<\theta} \chi_\zeta / I)\Big\}.$$

For simplicity assume that for $\chi < \mu$ and $A \subseteq (2^\chi)^+$, in $K[A]$ there are Ramsey cardinals $> \chi$. This makes a minor restriction; say for one $\lambda$ we may get $\leq \lambda^+$ instead of $< \lambda^+$ (which is equivalent to $< \lambda$).

So by [Sh 430, §5], for some uncountable regular $\sigma < \kappa$ from $R \setminus \mathrm{cf}(\mu)^+, \oplus^\sigma_{\mu,\lambda}$ from the assumption of the theorem holds and for some family $E$ of ideals on $\kappa$ normal by a function $\imath\colon \kappa \to \sigma$ and $J \in E$ and $\lambda_i = \mathrm{cf}(\lambda_i) \in (\mu, \lambda)$, $\lambda^+ = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i / J)$ and $\langle \lambda_i\colon i < \kappa\rangle$, $J$ minimal in a suitable sense, that is $\alpha(*) = \mathrm{rk}^3_J(\langle \lambda_i\colon i < \kappa\rangle, E)$ is minimal, so without loss of generality $\mathrm{rk}^3_J(\langle \lambda_i\colon i < \kappa\rangle, E) = \mathrm{rk}^2_J(\langle \lambda_i\colon i < \kappa\rangle, E)$. Hence we do not have $A \subseteq \kappa$, $\kappa \setminus A \notin J$ and $\lambda'_i \in (\mu, \lambda) \cap$ Reg such that $\langle \lambda'_i\colon i < \kappa\rangle <_{J+A} \langle \lambda_i\colon i < \kappa\rangle$ and $\lambda^+ = \mathrm{tcf}(\prod_{i<\kappa} \lambda'_i / J)$. As $\mathrm{cf}(\mu) = \aleph_0$, we can find $\langle \theta_n\colon n < \omega\rangle, \kappa < \theta_n \in R \cap \mu$ and $\mu = \bigcup_{n<\omega} \theta_n$. As $\lambda$ is minimal there is a partition $\langle u(n)\colon n < \omega\rangle$ of $\kappa$, such that:

$$(*) \qquad i \in u(n),\ n < \omega,\ |\mathfrak{a}| < \mu,\ \mathfrak{a} \subseteq \text{ Reg } \cap (\mu, \lambda_i) \Rightarrow \mathrm{pcf}_{\theta_n\text{-complete}}(\mathfrak{a}) \subseteq \lambda_i.$$

So for some $n$ we have $u(n) \in J^+$. Without loss of generality $(\forall i < \kappa)(\lambda_i > \mu^+)$ and (as $\sigma > \aleph_0$) for some $n = n(*)$ we have $u(n) = \kappa$ (i.e., the minimality of $\alpha(*)$ is preserved). Choose $\theta \in R \cap \mu$ large enough such that

$$(\forall \mathfrak{a})\big[\mathfrak{a} \subseteq \text{ Reg } \cap (\mu, \lambda) \text{ and } |\mathfrak{a}| \leq \theta_{n(*)} + \kappa \Rightarrow |\lambda \cap \mathrm{pcf}_{\sigma\text{-complete}}(\mathfrak{a})| < \theta\big].$$

(Why is this possible? As $\oplus^\sigma_{\mu,\lambda}$ which holds by the choice of $\sigma$.) As $\theta \in R \cap \mu$ we can choose a sequence $\langle \chi_\zeta\colon \zeta < \theta\rangle$ and $I \supseteq J^{\mathrm{bd}}_\theta$ a $\theta$-complete ideal on $\theta$ such that $\chi_\zeta \in (\mu, \lambda)$ and $\mathrm{tcf}(\prod_{\zeta<\theta} \chi_\zeta / I) = \lambda^+$. By [Sh:g, Ch. IX, 4.1] we can find $\tau^\zeta_i = \mathrm{cf}(\tau^\zeta_i) \in (\mu, \lambda_i)$, $\tau^\zeta_i < \lambda_i$ such that $\chi_\zeta = \mathrm{tcf}(\prod_{i<\kappa} \tau^\zeta_i / J)$.

Now $\mathfrak{a} =: \lambda \cap \mathrm{pcf}_{\sigma\text{-complete}}\{\tau^\zeta_i\colon i < \kappa, \zeta < \theta\}$ has cardinality $< \mu$ (by the choice of $\sigma$) and has a smooth closed representation $\langle \mathfrak{b}_\Upsilon(\mathfrak{a})\colon \Upsilon \in \mathfrak{a}\rangle$ (see [Sh 430, 6.7]). For $i < \kappa$ there is $\mathfrak{c}_i \subseteq \mathrm{pcf}_{\theta_{n(*)}\text{-complete}}\{\tau^\zeta_i\colon \zeta < \theta\}$ such that $|\mathfrak{c}_i| < \theta_{n(*)}$ and $\bigwedge_{\zeta<\theta} \tau^\zeta_i \in \cup\{\mathfrak{b}_\Upsilon(\mathfrak{a}) : \Upsilon \in \mathfrak{c}_i\}$ (by the choice of $n(*)$ and by [Sh 430, 6.7], note that $\sigma < \kappa < \theta_{n(*)}$ by their choices, hence $\mathrm{pcf}_{\theta_{n(*)}\text{-complete}}\{\tau^\zeta_i\colon \zeta < \theta\} \subseteq \mathfrak{a}$ hence all is O.K.). Also $\mathfrak{c}_i \subseteq \lambda_i$ because we are assuming $u_{n(*)} = \kappa$.

Let

$$\mathfrak{d} =: \Big\{ \mathrm{tcf}(\prod_{i \in A} \tau_i/(J+A)) \colon \langle \tau_i : i < \kappa \rangle \in \prod_{i<\kappa} \mathfrak{c}_i \text{ and } A \in J^+ \text{ and } \mathrm{tcf}(\prod_{i \in A} \tau_i/(J+A)) \text{ is well defined} \Big\}.$$

Let $\mathfrak{c} = \bigcup_{i<\kappa} \mathfrak{c}_i$. So $|\mathfrak{c}| \le \kappa + \theta_{n(*)}$ hence $\lambda \cap \mathrm{pcf}_{\sigma\text{-complete}}(\mathfrak{c})$ has cardinality $< \theta$, and $\mathfrak{d} \subseteq \lambda$ by the choice of $\alpha(*)$ and $\mathfrak{d} \subseteq \mathrm{pcf}_{\sigma\text{-complete}}(\mathfrak{c})$ hence $|\mathfrak{d}| < \theta$ (by the choice of $\theta$).

Now if $\psi \in \lambda^+ \cap \mathrm{pcf}(\mathfrak{c})$ then

$$B_\psi = \{\zeta < \theta \colon \{i < \kappa \colon \tau_i^\zeta \in \mathfrak{b}_\psi[\mathfrak{a}]\} \notin J\} \in I.$$

[Why? Otherwise $\zeta \in B_\psi \Rightarrow \chi_\zeta \in \mathrm{pcf}(\mathfrak{b}_\psi[\mathfrak{a}])$ hence $\mathrm{pcf}(\mathfrak{b}_\psi[\mathfrak{a}])$ includes $\mathrm{pcf}\{\chi_\zeta \colon \zeta \in B_\psi\}$, but as $B_\psi \notin I$ the cardinal $\lambda^+$ belongs to the latter; but $\max \mathrm{pcf}(\mathfrak{b}_\psi[\mathfrak{a}]) = \psi < \lambda$, contradiction.]

But we know that $|\mathfrak{d}| < \theta$, and $I$ is $\theta$-complete and $\mathfrak{d} \subseteq \mathrm{pcf}(\mathfrak{c})$, so

$$X = \Big\{\zeta < \theta \colon \text{ for some } \psi \in \mathfrak{d} \text{ we have } \{i < \kappa \colon \tau_i^\zeta \in \mathfrak{b}_\psi[\mathfrak{a}]\} \notin J \Big\} \subseteq \bigcup_{\psi \in \mathfrak{d}} B_\psi \in I.$$

So there is some $\zeta^* \in \theta \setminus X$, and for $i < \kappa$ choose $\Upsilon_i \in \mathfrak{c}_i$ such that $\tau_i^{\zeta^*} \in \mathfrak{b}_{\Upsilon_i}[\mathfrak{a}]$ (well defined by the choice of $\mathfrak{c}_i$). So by smoothness of the representation

$$\psi \in \mathfrak{d} \Rightarrow \{i < \kappa \colon \Upsilon_i \in \mathfrak{b}_\psi[\mathfrak{a}]\} \subseteq \{i < \kappa \colon \tau_i^{\zeta^*} \in \mathfrak{b}_\psi[\mathfrak{a}]\} \in J.$$

Now by the pcf theorem for some $A \in J^+$ we have $\prod_{i \in A} \Upsilon_i^{\zeta^*}/J$ has true cofinality which we call $\Upsilon$, so necessarily $\Upsilon \in \mathrm{pcf}_{\sigma\text{-complete}}(\{\Upsilon_i^{\zeta^*} : i \in A\}) \in \mathfrak{d}$ (see the definition of $\mathfrak{d}$), but this contradicts the previous sentence (recall $\mathfrak{d} \subseteq \lambda$ by the minimality of $\alpha(*)$). $\blacksquare_{2.10}$

## 3. Applications

Of course

3.1 Claim: *If $\mu$ is as in 2.1, then the conclusions of 1.2 and 1.1 hold.*

3.2 CLAIM: *If* $\lambda \geq \beth_\omega$ *then:*

(a) $2^\lambda = \lambda^+ \Leftrightarrow \diamondsuit_{\lambda^+}$,

(b) $\lambda = \lambda^{<\lambda}$ *iff* $(D\ell)_\lambda$;

where we remember

*3.3 Definition:* 1) $(D\ell)_\lambda$ means that:

$\lambda$ is regular uncountable and there is $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha \colon \alpha < \lambda \rangle$ such that $\mathcal{P}_\alpha$ is a family of $< \lambda$ subsets of $\alpha$ satisfying:

($*$) for every $A \subseteq \lambda$, $\{\alpha < \lambda \colon A \cap \alpha \in \mathcal{P}_\alpha\}$ is a stationary subset of $\lambda$.

2) $(D\ell)^*_S$ ($S \subseteq \lambda$ stationary) means $\lambda$ regular and there is $\bar{\mathcal{P}}$ as above such that:

($*$) for every $A \subseteq \lambda$ we have $\{\alpha \in S \colon A \cap x \notin \mathcal{P}_\alpha\}$ is not stationary.

3) $(D\ell)^+_S$ where $S \subseteq \lambda$ is stationary, $\lambda$ regular uncountable means that: for some $\bar{\mathcal{P}}$ as above:

($*$) for every $A \subseteq \lambda$ for some club $C$ of $\lambda$ we have:

$\delta \in S \cap C \Rightarrow A \cap \delta \in \mathcal{P}_\delta$ & $C \cap \delta \in \mathcal{P}_\delta$.

4) Let $\lambda$ be regular uncountable, $S \subseteq \lambda$ stationary. Now $\diamondsuit_S$ means that there is $\langle A_\alpha \colon \alpha \in S \rangle$ such that $A_\alpha \subseteq \alpha$ and for every $A \subseteq \lambda$ the set $\{\alpha \in S \colon A \cap \alpha = A_\alpha\}$ is a stationary subset of $\lambda$.

5) For $\lambda$ regular uncountable and $S \subseteq \lambda$ stationary $(D\ell)_S$ means that for some $\langle \mathcal{P}_\alpha \colon \alpha \in S \rangle$ as above, for every $A \subseteq \lambda$ the set $\{\delta \in S \colon A \cap \delta \in \mathcal{P}_\delta\}$ is stationary.

*3.4 Remark:* 1) If $\lambda$ is a successor cardinal, $(D\ell)_\lambda$ is equivalent to $\diamondsuit_\lambda$ (by Kunen), so (a) is a particular case of (b) in 3.2.

2) By [Sh 82], [HLSh 162], if $(D\ell)_\lambda$ then the omitting types theorem for $L(Q)$ for $\lambda$-compact models in the $\lambda^+$-interpretation holds (and more). Now $\lambda = \lambda^{<\lambda}$ is the standard assumption to the completeness theorem of $L(Q)$ in the $\lambda^+$-interpretation; and is necessary and sufficient when we restrict ourselves to $\lambda$-compact models. So the question arises, how strong is this extra assumption? If G.C.H. holds $(D\ell)_\lambda \Leftrightarrow \lambda = \lambda^{<\lambda}$ for every $\lambda \neq \aleph_1$ (by [Sh:82], continuing Gregory [Gre]); and more there. Here we improve those theorems. Now 3.2 says that above $\beth_\omega$, the two conditions are equivalent.

3) We may consider the function $h : \lambda \to \lambda \cap$ Car, demanding $|\mathcal{P}_\alpha| < h(\alpha)$.

4) Remember that for $\lambda > \aleph_0$ regular and stationary $S = S_1 \subseteq S_2 \subseteq \lambda$ we have $(D\ell)^+_S \Rightarrow (D\ell)^*_S \Rightarrow (D\ell)_S$ and $(D\ell)_{S_1} \Rightarrow (D\ell)_{S_2}$, but $(D\ell)^*_S \Rightarrow (D\ell)^*_{S_1}, (D\ell)^+_{S_2} \Rightarrow (D\ell)^+_{S_1}$.

*3.5 Proof of 3.2:* By 3.4(1) it suffices to prove clause (b). Trivially $(D\ell)_\lambda \Rightarrow \lambda = \lambda^{<\lambda}$, so assume $\lambda = \lambda^{<\lambda}$, and let $\{A^*_i \colon i < \lambda\}$ list the bounded subsets of $\lambda$, each appearing $\lambda$ times.

For each $\alpha < \lambda$ let

$$R_\alpha = \{\kappa < \beth_\omega\colon\ \mathrm{cov}(|\alpha|, \kappa^+, \kappa^+, \kappa) < \lambda \text{ and } \kappa \text{ is regular}\}.$$

We know (by 1.2(3)) that for each $\alpha \in (\beth_\omega, \lambda)$, $R_\alpha$ contains a co-bounded subset of Reg $\cap \beth_\omega$, say Reg $\cap \beth_\omega \setminus \beth_{n_\alpha}$. So for some $n^* < \omega$

$$S^* = \{\alpha < \lambda\colon \alpha > \beth_\omega,\ n_\alpha < n^*\}$$

is unbounded in $\lambda$; hence trivially $S^* = (\beth_\omega, \lambda)$. So $R =: \{\kappa < \lambda\colon \kappa$ is regular, $2^\kappa < \lambda$ and for every $\alpha < \lambda$ we have $\mathrm{cov}(|\alpha|, \kappa^+, \kappa^+, \kappa) < \lambda\}$ contains Reg $\cap$ $(\beth_{n^*}, \beth_\omega)$. As $\lambda = \mathrm{cf}(\lambda) > \beth_\omega$, for each $\alpha < \lambda$, $\kappa \in R$ there is $\mathcal{P}^\kappa_\alpha$, a family of $< \lambda$ subsets of $\alpha$ of cardinality $\kappa$ such that if $A \subseteq \alpha$, $|A| = \kappa$ then $A$ is included in the union of $< \kappa$ members of $\mathcal{P}^\kappa_\alpha$.

Let $\mathcal{P}^*_\alpha = \{B\colon$ for some $\kappa \in R \cap (\alpha + 1)$ and $A \in \mathcal{P}^\kappa_\alpha$ we have $B \subseteq A\}$ so $\mathcal{P}^*_\alpha$ is a family of $< \lambda$ subsets of $\alpha$. For each $A \subseteq \lambda$ we define $h_A\colon \lambda \to \lambda$ by defining $h_A(\alpha)$ by induction on $\alpha$ : for $\alpha$ non-limit $h_A(\alpha)$ is the first ordinal $i > \bigcup_{\beta<\alpha} h_A(\beta) + 1$ such that $A \cap \alpha = A^*_i$ and for $\alpha$ limit $h_A(\alpha) = \bigcup_{\beta<\alpha} h_A(\beta)$. So $h_A(\alpha)$ is strictly increasing continuous, hence $h_A(\alpha) \geq \alpha$ and $h(\alpha) = \alpha \leftrightarrow [(\alpha \text{ limit}) \ \&\ (\forall \beta < \alpha)(h_A(\beta) < \alpha)]$. Let

$$\mathcal{P}^0_\alpha =: \left\{ \bigcup_{\beta \in B} A^*_\beta\colon B \in \mathcal{P}^*_\alpha \right\},$$

$$\mathcal{P}_\alpha =: \mathcal{P}^0_\alpha \cup \left\{ \{\beta < \alpha\colon h_A(\beta) = \beta\}\colon A \in \mathcal{P}^0_\alpha \right\}$$

(remember $\langle A^*_\alpha\colon \alpha < \lambda\rangle$ lists the bounded subsets of $\lambda$ each appearing unboundedly often).

Now for any $A \subseteq \lambda$ we have $E =: E_A =: \{\delta < \lambda\colon \delta$ limit and $\bigwedge_{\beta<\delta} h_A(\beta) < \delta\}$ is a club of $\lambda$, and

$(*)_1$ $\mathrm{cf}(\delta) < \delta \in E$ & $\mathrm{cf}(\delta) \in R \Rightarrow A \cap \delta \in \mathcal{P}^0_\delta \subseteq \mathcal{P}_\delta$.

[Why? Let $\kappa =: \mathrm{cf}(\delta)$, and let $\langle \beta_j\colon j < \kappa\rangle$ be an increasing sequence of successor ordinals with limit $\delta$, hence $\langle h_A(\beta_j)\colon j < \kappa\rangle$ is (strictly) increasing with limit $\delta$; so for some $\beta < \kappa = \mathrm{cf}(\delta)$ and $B_i \in \mathcal{P}^{\mathrm{cf}(\delta)}_\delta$ for $i < \beta$ we have $\{h_A(\beta_j)\colon j < \kappa\} \subseteq \bigcup_{i<\beta} B_i$, so for some $i$, $\{h_A(\beta_j)\colon j < \kappa, h_A(\beta_j) \in B_i\}$ is unbounded in $\delta$, and clearly $B' =: \{h_A(\beta_j)\colon j < \kappa\} \cap B_i \in \mathcal{P}^*_\delta$, hence $\cup\{A^*_\gamma\colon \gamma \in B'\} \in \mathcal{P}^0_\alpha$ is as required].

Also

$(*)_2$ $\mathrm{cf}(\delta) < \delta \in E$ and $\mathrm{cf}(\delta) \in R \Rightarrow E \cap \delta \in \mathcal{P}$.

[Why? As $A \subseteq \lambda$, $\delta \in E_A \Rightarrow h_A \restriction \delta = h_{A\cap\delta} \restriction \delta$.] $\blacksquare_{3.2}$

Note that we actually proved also

3.6 Claim: *1) Assume $\lambda = \mu^+ = 2^\mu > \chi$, $\chi$ strong limit, then for some $\chi^* < \chi$ we have $\diamondsuit^+_{\{\delta<\lambda:\ \chi^*<\mathrm{cf}(\delta)<\chi\}}$.*
*2) Similarly, for $\lambda = \lambda^{<\lambda}$ inaccessible, $\chi$ strong limit $< \lambda$ for some $\chi^* < \chi$, $(D\ell)^+_{\{\delta<\lambda:\ \chi^*<\ \mathrm{cf}(\delta)<\chi\}}$ holds.*
*3) If $\lambda = \lambda^{<\lambda}$, and*

$$S = \{\delta < \lambda : \mathrm{cf}(\delta) < \delta, 2^{\mathrm{cf}(\delta)} < \lambda,\ \text{and}\ [\lambda > \mathrm{cov}(|\delta|, \mathrm{cf}(\delta)^+, \mathrm{cf}(\delta)^+, \mathrm{cf}(\delta)]\}$$

*then $(D\ell)^+_S$; so if $\lambda$ is a successor cardinal we have $\diamondsuit^+_S$.*
*4) Assume*[4] *$\lambda = \lambda^{<\lambda} > \theta = \mathrm{cf}(\theta) > \sigma = \mathrm{cf}(\sigma)$, $\theta^\sigma < \lambda$, $\theta^+ < \lambda$, $S \subseteq \lambda$, $\{\delta \in S$: $\mathrm{cf}(\delta) = \theta\}$ is stationary, $\bar{C} = \langle C_\alpha$: $\alpha \in S\rangle$, for $\alpha \in S$, $C_\alpha$ is a closed subset of $\alpha$, $[\beta \in C_\alpha \Rightarrow \beta \in S\ \&\ C_\beta = \beta \cap C_\alpha]$. Assume further that for no $\alpha < \lambda$ is there $\mathcal{P} \subseteq \{a \subseteq \alpha$: $|a| = \theta\}$, such that $[a \in \mathcal{P}\ \&\ b \in \mathcal{P}\ \&\ a \neq b \Rightarrow |a \cap b| < \sigma]$, and $[\mathfrak{a} \subseteq \lambda \cap \mathrm{Reg} \setminus \mathrm{Min}\,\mathfrak{a} > \theta\ \&\ |\mathfrak{a}| < \theta \Rightarrow \lambda > \sup(\lambda \cap \mathrm{pcf}\mathfrak{a})]$ (e.g., $\lambda$ successor).*
*Then $(D\ell)_{S_\sigma}$ holds where $S_\sigma = \{\delta \in S : \mathrm{cf}(\delta) = \sigma\}$.*

*Proof:* Easy. For example, 4) By [Sh:g, Ch. III, §2], without loss of generality for every club $E$ of $\lambda$ for some $\delta \in E \cap S$, $C_\delta \subseteq E$ if $\theta^+ < \lambda$, and otp $(C_\delta \cap E) < \theta$ otherwise. Let $\chi = \beth_3(\lambda)^+$, let $\langle M_i$: $i < \lambda\rangle$ be such that: $M_i \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, $\|M_i\| < \lambda$, $\lambda \in M_i$, $M_i \cap \lambda$ an ordinal, $\langle M_j$: $j \leq i\rangle \in M_{i+1}$. Let for $\delta \in S_\sigma$, $\mathcal{P}_\delta = M_{\delta+1} \cap \mathcal{P}(\delta)$. It is enough to show that $\bar{\mathcal{P}} = \langle \mathcal{P}_\delta$: $\delta \in S_\sigma\rangle$ exemplifies $(D\ell)_{S_\sigma}$. So let $\langle x_\alpha$: $\alpha < \lambda\rangle \in M_0$ list the bounded subsets of $\lambda$ each appearing $\lambda$ times. Let $X \subseteq \lambda$, $E_0$ be a club of $\lambda$; we define by induction on $\alpha$, $h_X(\alpha) < \lambda$ as the first $\gamma < \lambda$ such that $\gamma > \bigcup_{\beta<\alpha} h_X(\beta)$ and $X \cap \alpha = X_\alpha$. Let $\langle M^*_i$: $i < \lambda\rangle$ be chosen as above but also $h_X \in M^*_0$, $\langle M_i$: $i < \lambda\rangle \in M^*_0$, $E_0 \in M^*_0$. Let $E =: \{\delta \in E_0$: $M^*_\delta \cap \lambda = \delta = M_\delta \cap \lambda\}$; clearly it is a club of $\lambda$. Let $\delta \in S \cap E$, $\mathrm{cf}(\delta) = \theta$ be such that $C_\delta \subseteq E$. Now we imitate the proof [Sh 410, §6] or directly as in [Sh 420, §1] for $h_X \restriction C_\delta$. $\blacksquare_{3.6}$

3.7 Claim: *Above, instead of demanding on $\kappa$*

$$\text{“}\kappa = \mathrm{cf}(\kappa)\ \&\ 2^\kappa < \lambda\ \&\ [\alpha < \lambda \Rightarrow \mathrm{cov}(|\alpha|, \kappa^+, \kappa^+, \kappa) < \lambda]\text{”}$$

*it suffices to demand “$\kappa = \mathrm{cf}(\kappa) < \lambda$ and if $T$ is a tree with $\kappa$-levels and $< \lambda$ nodes then $T$ has $< \lambda$ $\kappa$-branches”. See* [Sh 589, §2] *for a pcf-characterization of this property.*

---

4 If $\lambda = \mu^+$, $\mu = \mathrm{cf}(\mu) > \theta = \mathrm{cf}(\theta) > \sigma = \mathrm{cf}(\sigma)$ then there are $S, \bar{C}$ as in 3.6(4)(see [Sh 351, §4] or [Sh:g, Ch. III, 2.14]+[E12]). Of course, we get not just guessing on a stationary set but on a positive set modulo a larger ideal.

3.8 LEMMA: (1) *Suppose $c\ell$ is an operation on $X$, i.e., $c\ell$ is a function from $\mathcal{P}(X)$ to $\mathcal{P}(X)$. Assume further $\kappa \le \kappa^* < \mu = \mu^\kappa$ and we let*

$$\mathcal{P}^* = \Big\{ A \subseteq X: |A| = \mu \text{ and for every } B \subseteq A \text{ satisfying } |B| = \kappa^* \text{ there is } B' \subseteq B, |B'| = \kappa \text{ such that } c\ell(B') \subseteq A, \text{ and } |c\ell(B')| = \mu \Big\}.$$

*If $\kappa^* < \beth_\omega(\kappa) \le \mu$ then there is function $h: X \to \mu$ such that: if $A \in \mathcal{P}^*$ then $h \restriction A$ is onto $\mu$.*

*(2) Actually, instead of "$\beth_\omega(\kappa) \le \mu$" we just need a conclusion of it:*

$$(*)_1 = (*)^1_{\mu,\kappa^*} \quad (\forall \lambda \ge \mu)(\exists \theta)[\theta \in \text{Reg and } \kappa^* \le \theta \le \mu \text{ and } \text{cov}(\lambda, \theta^+, \theta^+, \theta) = \lambda],$$

*or even just a conclusion of that:*

$(*)_2 = (*)^2_{\mu,\kappa^*}$ *for every $\lambda \ge \mu$ for some $\theta < \mu$, $\theta \ge \kappa^*$ we have:*

*$\otimes^\theta_\lambda = \otimes^{\theta,\kappa^*}_\lambda$: there is no family $\mathcal{P}$ of $> \lambda$ subsets of $\lambda$ each of cardinality $\theta$ with the intersection of any two having cardinality $< \kappa^*$.*

*3.9 Remark:* (1) The holding of $(*)_2$ is characterized in [Sh 410, §6].

(2) On earlier results concerning such problems and earlier history see Hajnal, Juhász and Shelah [HJSh 249]. In particular, the following is quite a well known problem:

$\bigoplus$ Non-compactum partition problem: Can every topological space be divided into two pieces, such that no part contains a closed homeomorphic copy of ${}^\omega 2$ (or any topological space $Y$ such that every scattered set is countable, and the closure of a non-scattered set has cardinality continuum)?

(3) Note that the condition in $(*)_2$ holds if $\mu = 2^{\aleph_0} > \aleph_\omega$, $\kappa = \aleph_0$, $\kappa^* = \aleph_1$ and $\otimes^1_{\aleph_\omega}$ (from 2.1) (which holds, e.g., if $V = V_0^P$, $P$ a c.c.c. forcing making the continuum $> \beth_\omega^{V_0}$). So in this case the answer to $\oplus$ is positive.

(4) Also if $\mu = 2^{\aleph_0} > \theta \ge \aleph_1$, and $(\forall \lambda)[\lambda \ge 2^{\aleph_0} \Rightarrow \otimes^{\theta,\aleph_1}_\lambda]$ then the answer to $\oplus$ in (2) is yes; now on $\otimes^{\theta,\aleph_1}_\lambda$ see [Sh 410, §6].

*Proof:* We prove by induction on $\lambda \in [\mu, |X|]$ that:

$(*)_\lambda$ if $Z, Y$ are disjoint subsets of $X$, $|Y| \le \lambda$, then there is a set $Y^+$, $Y \subseteq Y^+ \subseteq X \setminus Z$, $|Y^+| \le \lambda$ and a function $h: Y^+ \to \mu$ such that: if $A \in \mathcal{P}^*$, $\kappa^* \le \theta < \mu$, $\otimes^\theta_\lambda$, $|A \cap Y^+| \ge \theta$ and $|A \cap Z| < \mu$ then $h \restriction (A \cap Y^+)$ is onto $\mu$.

CASE 1: $\lambda = \mu$, so $|Y| \le \mu$.

Without loss of generality $[B \subseteq Y$ and $|B| \le \kappa$ and $|c\ell(B)| = \mu \Rightarrow c\ell(B) \setminus Z \subseteq Y]$. Now just note that $\mathcal{P}_Y =: \{c\ell(B) \cap Y \colon B \subseteq Y, |B| \le \kappa, |c\ell(B) \cap Y| = \mu\}$ has cardinality $\le \mu = \mu^\kappa$, and by the definition of $\mathcal{P}^*$ (using the demand $|A \cap Z| < \mu$ in $(*)_\lambda$), it suffices that $h$ satisfies: $[A \in \mathcal{P}_Y \Rightarrow h \restriction Z$ is onto $\mu]$, which is easily accomplished.

CASE 2: $\lambda > \mu$.

Let $\chi = (2^\lambda)^+$, $\langle N_i \colon i \le \lambda \rangle$ an increasing continuous sequence of elementary submodels of $(\mathcal{H}(\chi), \in, <^*_\chi)$, $\langle X, c\ell, Y, Z, \lambda \rangle \in N_0$, $\mu+1 \subseteq N_0$, $\langle N_i : i \le j \rangle \in N_{j+1}$ (when $i < \lambda$) and $\|N_i\| = \mu + |i|$.

We define, by induction on $i < \lambda$, a set $Y_i^+$ and a function $h_i$ as follows:

$(Y_i^+, h_i)$ is the $<^*_\chi$-first pair $(Y^*, h^*)$ such that:

(a) $Y^* \subseteq X \setminus (Z \cup \bigcup_{j<i} Y_j^+)$,

(b) $Y \cap N_i \setminus \bigcup_{j<i} Y_j^+ \setminus Z \subseteq X \cap N_i \setminus \bigcup_{j<i} Y_j^+ \setminus Z \subseteq Y^*$,

(c) $|Y^*| = \mu + |i|$,

(d) $h^* \colon Y^* \to \mu$,

(e) if $A \in \mathcal{P}^*$, $\otimes^\theta_{\mu+|i|}$, $\kappa^* \le \theta < \mu$, $|A \cap Y^*| \ge \theta$, $|A \cap (Z \cup \bigcup_{j<i} Y_j^+)| < \mu$ then $h^* \restriction (A \cap Y^*)$ is onto $\mu$.

*Note*: $(Y_i^+, h_i)$ exists by the induction hypothesis applied to the cardinal $\mu + |i|$ and the sets $Z \cup \bigcup_{j<i} Y_j^+$, $X \cap N_i \setminus \bigcup_{j<i} Y_j^+$. Also, it is easy to check that $\langle (Y_j^+, h_j) \colon j \le i \rangle \in N_{i+1}$ (as we always choose "the $<^*_\chi$-first", hence $Y_i^+ \subseteq N_{i+1}$).

Let $Y^+ = \bigcup_{i<\lambda} Y_i^+$, $h = \bigcup_{i<\lambda} h_i$. Clearly $Y \subseteq \bigcup_{i<\lambda} N_i$, hence by requirement (b) clearly $Y \subseteq Y^+$ (and even $X \cap N_\lambda \setminus Z \subseteq Y^+$); by requirements (c) (and (a)) clearly $|Y^+| = \lambda$, by requirement (a) clearly $Y^+ \subseteq X \setminus Z$ and even $Y^+ = X \cap N_\lambda \setminus Z$.

By requirements (a) + (d), $h$ is a function from $Y^+$ to $\mu$. Now suppose $A \in \mathcal{P}^*$, $\otimes^\theta_\lambda$, $\kappa^* \le \theta < \mu$, $|A \cap Y^+| \ge \theta$, $|A \cap Z| < \mu$; we should prove "$h \restriction (A \cap Y^+)$ is onto $\mu$". So $|A \cap N_\lambda| \ge \theta$. Choose $(\delta^*, \theta^*)$ a pair such that:

(i) $\delta^* \le \lambda$,

(ii) $\otimes^{\theta^*}_{\mu+|\delta^*|}$, $\kappa^* \le \theta^* < \mu$,

(iii) $|A \cap N_{\delta^*}| \ge \mu$ or $\delta^* = \lambda$,

(iv) under (i) + (ii) + (iii), $\delta^*$ is minimal.

This pair is well defined as $(\lambda, \theta)$ satisfies requirement (i) + (ii) + (iii).

SUBCASE 1: $\delta^*$ is zero.

So $|Y_0^+ \cap A| \ge \theta^* \ge \kappa^*$, hence by the choice of $h_0$ we are done.

SUBCASE 2: $\delta^* = i+1$.

So $|A \cap N_i| < \mu$, hence $|A \cap \bigcup_{j<i} Y_j^+| < \mu$, hence $|A \cap (Z \cup \bigcup_{j<i} Y_j^+)| < \mu$. Clearly $\otimes^{\theta^*}_{\mu+|i|}$ holds (as $\mu+|i| = \mu+|\delta^*|$), so if $|A \cap Y_i^+| \geq \theta^*$ we are done by the choice of $h_i$; if not, $|A \cap (Z \cup \bigcup_{j<i+1} Y_j^+)| < \mu$ and $A \cap Y_{i+1}^+ \supseteq A \cap N_{i+1} = A \cap N_{\delta^*}$ has cardinality $\geq \theta^*$ (and $\otimes^{\theta^*}_{|Y_{i+1}^+|}$ holds) so we are done by the choice of $h_{i+1}$.

SUBCASE 3: $\delta^*$ limit $< \lambda$.

So for some $i < \delta^*$, $|A \cap N_i| \geq \theta^*$ [why? as $\theta^* < \mu < \lambda$]. Now in $N_{i+1}$ there is a maximal family $Q \subseteq [X \cap N_i]^{\theta^*}$ satisfying $[B_1 \neq B_2 \in Q \Rightarrow |B_1 \cap B_2| < \kappa^*]$, hence $|Q| \leq \mu + |\delta^*|$ and without loss of generality $Q \in N_{i+1}$, hence $Q \subseteq N_{\delta^*}$ so there is $B \in Q$, $B \in N_{\delta^*}$, $|B \cap A| \geq \kappa^*$; but $|B| = \theta^* < \mu = \mu^\kappa$ hence $[B' \in [B \cap A]^\kappa \Rightarrow B \cap A \in N_{\delta^*}]$. As $A \in \mathcal{P}^*$ there is $B' \in [B \cap A]^\kappa$ with $c\ell(B') \subseteq A$, $|c\ell(B')| = \mu$. Clearly $c\ell(B') \in N_{\delta^*}$, hence for some $j \in (i, \delta^*)$, $c\ell(B') \in N_j$, hence $c\ell(B') \subseteq X \cap N_j$. So $|A \cap N_j| \geq \mu$. By assumption for some $\theta' \in [\kappa^*, \mu)$, $\otimes^{\theta'}_{\mu+|j|}$, so $(j, \theta')$ contradicts the choice of $(\delta^*, \theta^*)$.

SUBCASE 4: $\delta^*$ limit $= \lambda$.

As $\lambda \in N_0$, there is a maximal family $Q \subseteq [\lambda]^{\theta^*}$ satisfying $[B_1 \neq B_2 \in Q \Rightarrow |B_1 \cap B_2| < \kappa^*]$ which belongs to $N_0$. By the assumption $(*)_2$, we know $|Q| \leq \lambda$. We define, by induction on $j \leq \lambda$, a one-to-one function $g_j$ from $N_j \cap X \setminus Z$ onto an initial segment of $\lambda$ increasing continuous in $j$, $g_j$ the $<^*_\chi$-first such function. So clearly $g_j \in N_{j+1}$ and let $Q' = \{g_\lambda(B)\colon B \in Q\}$ (i.e. $\{\{g_\lambda(x)\colon x \in B\}\colon B \in Q\}$; note: $g_\lambda$ is necessarily a one-to-one function from $N_\lambda \cap X \setminus Z$ onto $\lambda$). So for some $B \in Q'$, $|B' \cap A| \geq \kappa^*$, so as in subcase 3, for some $B' \in N_\lambda$, $B' \subseteq B \cap A$, $|B'| = \kappa, c\ell(B') \subseteq A, |c\ell(B')| = \mu$; so for some $i < \lambda$, $c\ell(B') \subseteq N_i$. But $|A \cap Z| < \mu$, so $|A \cap Y_i^+| = \mu$ and by assumption $(*)_2$, for some $\theta$, $\kappa^* \leq \theta < \mu$ we have $\otimes^{\theta}_{\mu+|i|}$, contradicting the choice of $(\delta^*, \theta^*)$ (i.e., minimality of $\delta^*$). $\blacksquare_{3.8}$

*3.10 Discussion:* (1) So if we return to the topological problem (see $\oplus$ of 3.9(2)), by 3.8 + 3.9(4), if $2^{\aleph_0} > \theta \geq \aleph_1$ we can try $\theta = \aleph_2$, $\kappa^* = \aleph_0$, $\kappa = \aleph_1$. So a negative answer to $\oplus$ (i.e., the consistency of a negative answer) is hard to come by: it implies that for some $\lambda$, $\neg\otimes^{\theta,\aleph_1}_\lambda$, a statement which, when $\theta > \aleph_1$, at present we do not know is consistent (but clearly it requires large cardinals).

(2) If we want $\mu = 2^{\aleph_0} = \aleph_2$, $\theta = \aleph_1 = \kappa^*$ we should consider a changed framework. We have a family $\mathfrak{I}$ of ideals on cardinals $\theta < \mu$ which are $\kappa$-based (i.e., if $A \in I^+$, $I \in \mathfrak{I}$ (similar to [HJSh 249]) then $\exists B \in [A]^\kappa (B \in I^+)$) and in

3.8 replace $\mathcal{P}^*$ by

$$\mathcal{P}^* = \mathcal{P}^*_{\mathfrak{I}} =: \Big\{ A \subseteq X \colon |A| = \mu \text{ and for every pairwise distinct } x_\alpha \in A \text{ for } \alpha < \theta \text{ we have } \{u \subseteq \theta \colon |c\ell\{x_\alpha \colon \alpha \in u\}| < \mu\} \text{ is included in some } I \in \mathfrak{I} \Big\},$$

and replace $(*)_2$ by

$(*)_3$ For every $\lambda \geq \mu$ assume

$$F \subseteq \{(\theta, I, f) \colon I \in \mathfrak{I},\ \theta = \operatorname{Dom}(I),\ f \colon \theta \to \lambda \text{ is one to one}\}$$

and if $(\theta_\ell, I_\ell, f_\ell) \in F$ for $\ell = 1, 2$ are distinct then $\{\alpha < \theta_2 \colon f_2(\alpha) \in \operatorname{Rang} f_1\} \in I_2$.

Then $|F| \leq \lambda$.

Note that the present $\mathcal{P}^*$ fits for dealing with $\oplus$ of 3.9(2) and repeating the proof of 3.8.

*3.11 Discussion of Consistency of no*: There are some restrictions on such theorems. Suppose

$(*)$ GCH and there is a stationary $S \subseteq \{\delta < \aleph_{\omega+1} \colon \operatorname{cf}(\delta) = \aleph_1\}$ and $\langle A_\delta \colon \delta \in S \rangle$ such that: $A_\delta \subseteq \delta = \sup A_\delta$, $\operatorname{otp}(A_\delta) = \omega_1$ and $\delta_1 \neq \delta_2 \Rightarrow |A_{\delta_1} \cap A_{\delta_2}| < \aleph_0$.

(This statement is consistent by [HJSh 249, 4.6, p. 384] which continues [Sh 108].) Now on $\aleph_{\omega_1}$ we define a closure operation:

$$\alpha \in c\ell(u) \Leftrightarrow (\exists \delta \in S)[\alpha \in A_\delta \ \&\ (u \cap A_\delta) \geq \aleph_0].$$

This certainly falls under the statement of 3.8(2) with $\kappa = \kappa^* = \aleph_0, \mu = \aleph_1$, except the pcf assumptions $(*)_1$ and $(*)_2$ fail. However, this is not a case of our theorem.

## 4. Appendix: Existence of tiny models

We deal now with a model theoretic problem, the existence of tiny models; we continue Laskowski, Pillay, and Rothmaler [LaPiRo]; our main result is in 4.6.

*4.1 Context:* Assume $T$ is a complete first order theory. Let $|T|$ be the number of first order formulas $\varphi(\bar{x})$, $\bar{x} = \langle x_\ell \colon \ell < n \rangle$, $n < \omega$, up to equivalence modulo $T$.

Assume $T$ is categorical in all cardinals $\chi > \lambda =: |T|$ and call a model $M$ of $T$ tiny if $\|M\| < \mu(= |T|)$. It is known that a $T$ with a tiny model satisfies exactly one of the following:

(a) $T$ is totally transcendental, trivial (i.e., any regular type is trivial),

(b) $T$ is not totally transcendental.

4.2 QUESTION: *For which $\mu < \lambda$ are there $T$, $|T| = \lambda$ (which is categorical in $\lambda^+$ and) with a tiny model of cardinality $\mu$?*

*4.3 Discussion:* By [LaPiRo] we can deal with just the following two cases (see [LaPiRo], 0.3, p. 386 and $387^{1-21}$ and 1.7, p. 390).

CASE A: $x = x$ is a minimal formula and its prime model consists of individual constants.

CASE B: $T$ is superstable not totally transcendental and is unidimensional, the formula $x = x$ is weakly minimal, regular types are trivial and its prime model consists of individual constants.

They proved: $(\forall\kappa)[\kappa^{\aleph_0} \leq \kappa^+ \Rightarrow$ in case A, $\mu = \aleph_0]$ (see [LaPiRo, 2.1, p. 341]). Actually more is true by continuing their argument.

4.4 LEMMA: *If $\lambda, \mu, T$ are as above, in Case A, then:*

(i) $\lambda < \beth_\omega$,

(ii) *we can find $\langle \lambda_n : n < \omega \rangle$ such that: $\lambda_0 = \mu, \lambda_n \leq \lambda_{n+1}$, $\lambda = \sum_{n<\omega} \lambda_n$ and $(*)_{\mu,\lambda_n,\lambda_{n+1}}$*
*(hence in particular $(*)_{\mu,\mu,\mu^+}$), where*

$(*)_{\mu,\sigma,\theta}$ *there is a family of $\theta$ subsets of $\sigma$ each of cardinality $\mu$, with the intersection of any two being finite, or equivalently $\theta$ functions from $\mu$ to $\sigma$ such that for any two such distinct functions $f', f''$ we have $\{i < \mu : f'(i) = f''(i)\}$ is finite.*

*Proof:* By 1.2(2), (ii) $\Rightarrow$ (i), so let us prove (ii). Let $M$ be a tiny model of $T$, $\|M\| = \mu$.

For $n \geq 0$, let $\mathfrak{B}_n$ be the family of definable (with parameters) subsets of ${}^{n+1}M$. Clearly $|T| \leq \sum_{n<\omega} |\mathfrak{B}_n|$, also $\mu = \|M\| \leq |\mathfrak{B}_n|$, $|\mathfrak{B}_n| \leq |\mathfrak{B}_{n+1}|$. Also $|\mathfrak{B}_0| = \|M\|$ as $M$ is minimal which means $\lambda_0 = \mu$; let $\lambda_n =: |\mathfrak{B}_n|$, so $\lambda_n \leq \lambda_{n+1}$; $\mu = \sum_{n<\omega} \lambda_n$ and it is enough to prove $(*)_{\mu,\lambda_n,\lambda_{n+1}}$ when $\lambda_n < \lambda_{n+1}$. For each $R \in \mathfrak{B}_{n+1}$ we define a function $f_R$ from $M$ to $\mathfrak{B}_n$, $f_R(a) = \{\bar{b} \in {}^nM : \bar{b}\char`^ < a > \in R\}$. So $\{f_R : R \in \mathfrak{B}_{n+1}\}$ is a family of $\lambda_{n+1}$ functions from $M$ to $\mathfrak{B}_n$, hence it is enough to show:

$$\text{define } R_1 \approx R_2 \Rightarrow \{a \in M_0 : f_{R_1}(a) = f_{R_2}(a)\} \text{ is co-finite;}$$

then

($\alpha$) $\approx$ is an equivalence relation on $\mathfrak{B}_{n+1}$,

($\beta$) each $\approx$-equivalence class has cardinality $\leq \lambda_n$,

($\gamma$) if $\neg[R_1 \approx R_2]$, $R_1 \in \mathfrak{B}_{n+1}$, $R_2 \in \mathfrak{B}_{n+1}$ then $\{a \in M\colon f_{R_1}(a) = f_{R_2}(a)\}$ is finite.

Now clause ($\alpha$) is straight, for clause ($\beta$) just compute, for clause ($\gamma$) remember $x = x$ is a minimal formula. Together, a set of representations $\Upsilon$ for $\mathfrak{B}_{n+1}/\approx$ will have cardinality $\lambda_{n+1}$ (as $|\mathfrak{B}_{n+1}| = \lambda_{n+1} > \lambda_n = |\mathfrak{B}_n| \geq \mu$ by clauses ($\alpha$), ($\beta$)) and $\{f_R\colon R \in \Upsilon\}$ is a set of functions as required. $\blacksquare_{4.4}$

4.5 LEMMA: *Suppose $(*)_{\mu,\mu,\lambda}$, $\mu < \lambda$. Then*

(a) *there is a group $G$ of permutations of $\mu$ such that $|G| = \lambda$ and $f \neq g \in G \Rightarrow \{\alpha < \mu\colon f(\alpha) = g(\alpha)\}$ is finite,*

(b) *there is a theory $T$ as in 1.1, $|T| = \lambda$, with a tiny model of cardinality $\mu$ of Case A.*

*Proof:* As (a) $\Rightarrow$ (b) is proved in [LaPiRo], p. $392^{23-31}$ we concentrate on (a). Let pr$(-,-)$ be a pairing function on $\mu$, i.e., pr is one-to-one from $\mu \times \mu$ onto $\mu$. So let $\{A_\zeta\colon \zeta < \lambda\} \subseteq [\mu]^\mu$ be such that $\zeta \neq \xi \Rightarrow \aleph_0 > |A_\zeta \cap A_\xi|$. Clearly $\mu^{\aleph_0} \geq \lambda$, hence there is a list $\bar\eta\langle\eta_\zeta\colon \zeta < \lambda\rangle$ of distinct members of ${}^\omega\mu$. By renaming we can have the family $\{A_{\zeta,n}\colon \zeta < \lambda, n < \omega\}$, such that $(A_{\zeta,n} \in [\mu]^\mu, [(\zeta,n) \neq (\xi,m) \Rightarrow |A_{\zeta,n} \cap A_{\xi,m}| < \aleph_0]$ and) $\bigcup_{\zeta<\lambda} A_{\zeta,n} \cap \bigcup_{\zeta<\lambda} A_{\zeta,m} = \emptyset$ for $n \neq m$, and $\zeta \neq \xi \to (\exists n)(\forall m)[n \leq m < \omega \to A_{\zeta,n} \cap A_{\xi,n} = \emptyset]$ (use $\bar\eta()$. Let $g^0_{\zeta,n} \in {}^\mu\mu$ be $g^0_{\zeta,n}(\alpha) =$ the $\alpha$th member of $A_{\zeta,n}$ and $g^1_{\zeta,n}(\alpha) = pr(\alpha, g^0_{\zeta,n}(\alpha))$, so also $g^1_{\zeta,n}$ is a function from $\mu$ to $\mu$.

We define the set $A = \mu \times ({}^{\omega>}\{-1,+1\})$; clearly $|A| = \mu$. Let $x, y$ vary on $\{-1,+1\}$. Now for $\zeta < \lambda$ we define a permutation $f_\zeta$ of $A$, by defining $f^{+1}_\zeta \restriction (\mu \times \{\eta\}) = f_\zeta \restriction (\mu \times \{\eta\})$, $f^{-1}_\zeta \restriction (\mu \times \{\eta\})$ for $\eta \in {}^n\{-1,+1\}$ by induction on $n$ (so in the end, $f^{-1}_\zeta$ is the inverse of $f_\zeta = f^{+1}_\zeta$).

For $n = 0$, $\eta = \langle\rangle$ and let for $x \in \{-1,+1\}$, $f^x_\zeta(\alpha, \langle\rangle) = (g^1_{\zeta,0}(\alpha), \langle x\rangle)$.

For $n+1$, $\eta = \nu\char`^\langle y\rangle \in {}^{n+1}\{-1,+1\}$ we let

($\alpha$) $f^x_\zeta(\alpha,\eta) = (\beta,\nu)$ when
$x = -y, f^y_\zeta(\beta,\nu) = (\alpha,\eta)$ (by the previous stage),

($\beta$) $f^x_\zeta(\alpha,\eta) = (g^1_{\zeta,n+1}(\alpha), \eta\char`^\langle x\rangle)$ when ($\alpha$) does not apply.

Easily $f_\zeta$ is a well-defined permutation of $A$.

Now $\{f_\zeta : \zeta < \lambda\}$ generates a group $G$ of permutations of $A$. We shall prove it generates $G$ freely; moreover:

$\otimes$ if $n < \omega$, $t = \langle(\zeta(\ell), x(\ell)) : \ell \leq n\rangle$ is such that $\zeta(\ell) < \lambda$, $x(\ell) \in \{-1,1\}$, and for no $\ell < n$ do we have $\zeta(\ell) = \zeta(\ell+1)$ and $x(\ell) = -x(\ell+1)$

(i.e., $\prod_{\ell \le n} f^{x(\ell)}_{\zeta(\ell)}$ is a non-trivial group term) then

$$A_t = \{a \in A\colon (\prod_{\ell \le n} f^{x(\ell)}_{\zeta(\ell)})(a) = a\}$$

is finite.

As $|A| = \mu$, this clearly suffices.

As this property of $\prod_{\ell \le n} f^{x(\ell)}_{\zeta(\ell)}$ is preserved by conjugation without loss of generality

$(*)_0$ $\ell \le n \Rightarrow \zeta(\ell) \ne \zeta(\ell+1) \vee x(\ell) \ne x(\ell+1)$ where $n+1$ is interpreted as zero.

For any $a \in A_t$ let

$(*)_1$ $b^t_m[a] = (\prod^n_{\ell=m} f^{x(\ell)}_{\zeta(\ell)})(a)$ for $m \le n+1$
(so $b^t_{n+1}[a] = a = b^t_0[a]$ and for $m = 0, \dots, n$ we have

$$b^t_m[a] = f^{x(m)}_{\zeta(m)}(b^t_{m+1}[a])),$$

$(*)_2$ $b^t_m[a] = (\beta^t_m[a], \eta^t_m[a])$.

Choose $m^* < \omega$ large enough such that:

$(*)_3$ if $m \ge m^*$ and $0 \le \ell_1 < \ell_2 \le n$ and $\zeta(\ell_1) \ne \zeta(\ell_2)$ then $A_{\zeta(\ell_1),m} \cap A_{\zeta(\ell_2),m} = \emptyset$.

For $a \in A_t$ let $m = m[a] \le n+1$ be such that $lg(\eta^t_m[a])$ is maximal and call the length $k = k[a]$. As $f_\zeta(\langle \alpha, \eta \rangle) = \langle \beta, \nu \rangle$ implies $lg(\eta) \in \{lg(\nu) - 1, lg(\nu) + 1\}$, clearly

$(*)_4$ $lg(\eta^t_{m-1}[a]) = lg(\eta^t_{m+1}[a]) = lg(\eta^t_m[a]) - 1$ (where $m-1, m+1$ means mod $n+1$).

Clearly

$(*)_5$(a) $b^t_m[a] = f^{x(m)}_{\zeta(m)}(b^t_{m+1}[a])$,
(b) $b^t_{m-1}[a] = f^{x(m-1)}_{\zeta(m-1)}(b^t_m[a])$, hence (as $(f^{x(m-1)}_{\zeta(m-1)})^{-1} = f^{-x(m-1)}_{\zeta(m-1)}$) we have
(b)′ $b^t_m[a] = f^{-x(m-1)}_{\zeta(m-1)}(b^t_{m-1}[a])$.

Looking at the definition of $f^{-x(m-1)}_{\zeta(m-1)}(b^t_{m-1}[a])$, as $m = m[a]$, by $(*)_4$ clause $(\beta)$ in the definition of $f$ applies, so

$(*)_6$(a) $f^{-x(m-1)}_{\zeta(m-1)}(b^t_{m-1}[a]) = (g^1_{\zeta(m-1),k[a]}(\beta^t_{m-1}[a]), (\eta^t_{m-1}[a])\char`\^\langle -x(m-1)\rangle)$.

Similarly looking at the definition $f^{x(m)}_{\zeta(m)}(b^t_{m+1}[a])$, by $(*)_4$ clause $(\beta)$ applies, so

$(*)_6$(b) $f^{x(m)}_{\zeta(m)}(b^t_{m+1}[a]) = (g^1_{\zeta(m),k[a]}(\beta^t_{m+1}[a]), (\eta^t_{m+1}[a])\char`\^\langle x(m)\rangle)$.

By $(*)_5$(b)′ + $(*)_6$(a) we have

$(*)_7$(a) $b^t_m[a] = (g^1_{\zeta(m-1),k[a]}(\beta^t_{m-1}[a]), (\eta^t_{m-1}[a])\char`\^\langle -x(m-1)\rangle)$.

By $(*)_5$(a) + $(*)_6$(b) we have

$(*)_7$(b) $b^t_m[a] = (g^1_{\zeta(m),k[a]}(\beta^t_{m+1}[a]), (\eta^t_{m+1}[a])\char`\^\langle x(m)\rangle))$.

We can conclude by $(*)_7$(a) + $(*)_7$(b) that

$(*)_8$ $x(m) = -x(m-1)$, hence $x(m) \neq x(m-1)$.

So by $(*)_0$ applied to $m-1$ we get

$(*)_9$ $\zeta(m) \neq \zeta(m-1)$.

Clearly by $(*)_7$(a) + $(*)_7$(b) we have

$(*)_{10}$ $g^1_{\zeta(m),k[a]}(\beta^t_{m+1}[a]) = g^1_{\zeta(m-1),k[a]}(\beta^t_{m-1}[a])$.

Now by the choice of the $g^1_\zeta$'s (and the pairing function) and $(*)_{10}$

$(*)_{11}$ $\beta^t_{m+1}[a] = \beta^t_{m-1}[a]$ and $g^0_{\zeta(m),k[a]}(\beta^t_{m+1}[a]) = g^0_{\zeta(m-1),k[a]}(\beta^t_{m-1}[a])$.

So by $(*)_{11}$ and the choice of the $g^0_\zeta$'s

$(*)_{12}$ $g^0_{\zeta(m),k[a]}(\beta^t_{m+1}[a]) = g^0_{\zeta(m-1),k[a]}(\beta^t_{m-1}) \in A_{\zeta(m),k[a]} \cap A_{\zeta(m-1),k[a]}$.

If $k[a] > m^*$ we get a contradiction (by $(*)_3$), so remembering $m = m[a]$ necessarily $lg(\eta^t_{m[a]}[a]) \leq m^* + 1$, hence by the choice of $m[a]$ we have $\bigwedge_\ell lg(\eta^t_\ell[a]) \leq m^*$.

So $\{\langle \eta^t_\ell[a] \colon \ell < n+1\rangle \colon a \in A_t\}$ is finite, hence it suffices to prove for each $\bar\eta \in {}^{n+1}\{-1,1\}$ the finiteness of

$$A_{t,\bar\eta} = \{a \in A_t \colon \langle \eta^t_\ell[a] \colon \ell < n+1\rangle = \bar\eta\}.$$

Let us fix $\bar\eta$.

As for $a \in A_{t,\bar\eta}$ we have $\ell g(\eta^t_m[a]) \leq m^*$ for $\ell \leq n+1$, it is enough to prove that for each $\bar k = \langle k_\ell \colon \ell \leq n\rangle$ the following set is finite:

$$A_{t,\bar\eta,\bar k} =: \{a \in A_{t,\bar\eta} \colon \ell g(\eta^t_\ell[a]) = k_\ell \text{ for } < n+1\}.$$

Let $K(\bar k) = \{\ell \leq n+1 \colon k_\ell \text{ is } \geq k_{\ell-1}, k_{\ell+1}\}$ (i.e., a local maximum).

For each $m \in K(\bar k)$, the arguments in $(*)_3 - (*)_{12}$ apply, so by $(*)_{11}$, if $a \in A_{t,\bar\eta,\bar k}$ then the value $\ell g(\eta^t_m[a])$ is determined and $g^0_{\zeta(m),k_m}(\beta^t_{m+1}[a]) \in A_{\zeta(m),k_m} \cap A_{\zeta(m-1),k_m}$; but the latter is finite so we can fix $g^0_{\zeta(m),k_m}(\beta^t_{m+1}[a]) = \gamma_m$, but $g^1_{\zeta(m),k_m}(\beta^t_{m+1}[a])$ can be computed from $\gamma = g^0_{\zeta(m),k_m}(\beta^t_{m+1}[a])$ and $(\zeta(m), k_m)$, i.e., as $pr(\mathrm{otp}(A_{\zeta(m),k_m} \cap \gamma), \gamma_m)$.

But by $(*)_7$(b) the latter is $\beta^t_m[a]$ and as $\eta^t_m[a] = \eta_m$ the value of $b^t_m[a]$ is uniquely determined. Similarly by induction we can compute the other $b^t_{m'}[a]$ for every $m'$, in particular $b^t_0[a] = a$, so we are done. $\blacksquare_{4.5}$

*4.6 Conclusion:* For a cardinal $\mu$, the following are equivalent:

(a) there is a $T$ as in 4.5(b) (i.e., $T$ categorical in $|T|^+, |T| > \mu$), with a tiny model $M$, $\|M\| = \mu$ as in Case A above,

(b) $(*)_{\mu,\mu,\mu^+}$,

(c) there is a group $G$ of permutations of $\mu$, $|G| = \mu^+$ such that for $g \in G$, $\{\alpha < \mu \colon g(\alpha) = \alpha\}$ is finite or is $\mu$.

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